

# From Single Network to Network of Networks

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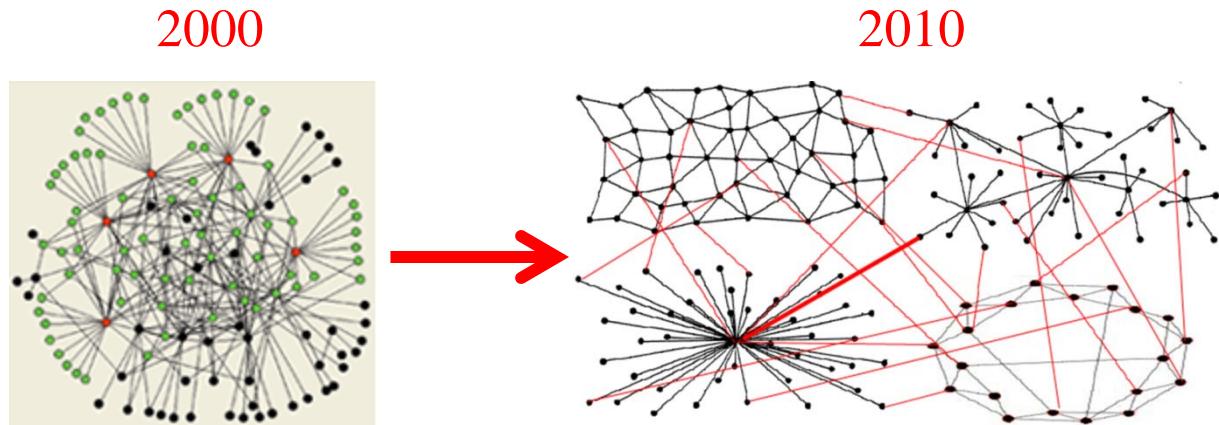
Dong Zhou, BIU

### PARTIAL LIST

Electric grid,  
Communication  
Transportation  
Services .....  
Protein interactions

- Two types of links:  
1. Connectivity  
2. Dependency (Albert)

Cascading failures-abrupt transition



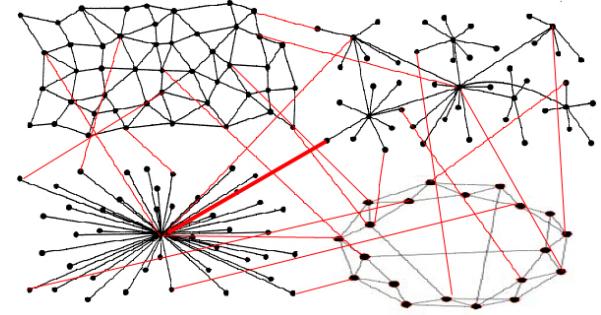
**Shlomo Havlin**

multilevel  
multilayer  
multiplex

- Buldyrev et al, Nature, 464, 1025 (2010 )  
Parshani et al, PRL ,105, 0484 (2010)  
Parshani et al, PNAS, 108, 1007 (2011)  
Gao et al, PRL, 107, 195701 (2011)  
Gao et al, Nature Phys.,8, 40 (2012)  
Wei Li et al, PRL, 108, 228702 (2012)  
Bashan et al, Nature Phys. 9, 667 (2013)  
Majdanzik et al Nature Phys. 10, 3438 (2014)  
Danziger et al, arXiv:1505.01688 (2015)  
Peixoto and Bornholdt, PRL, 109,118703(2012)

# Interdependent Networks

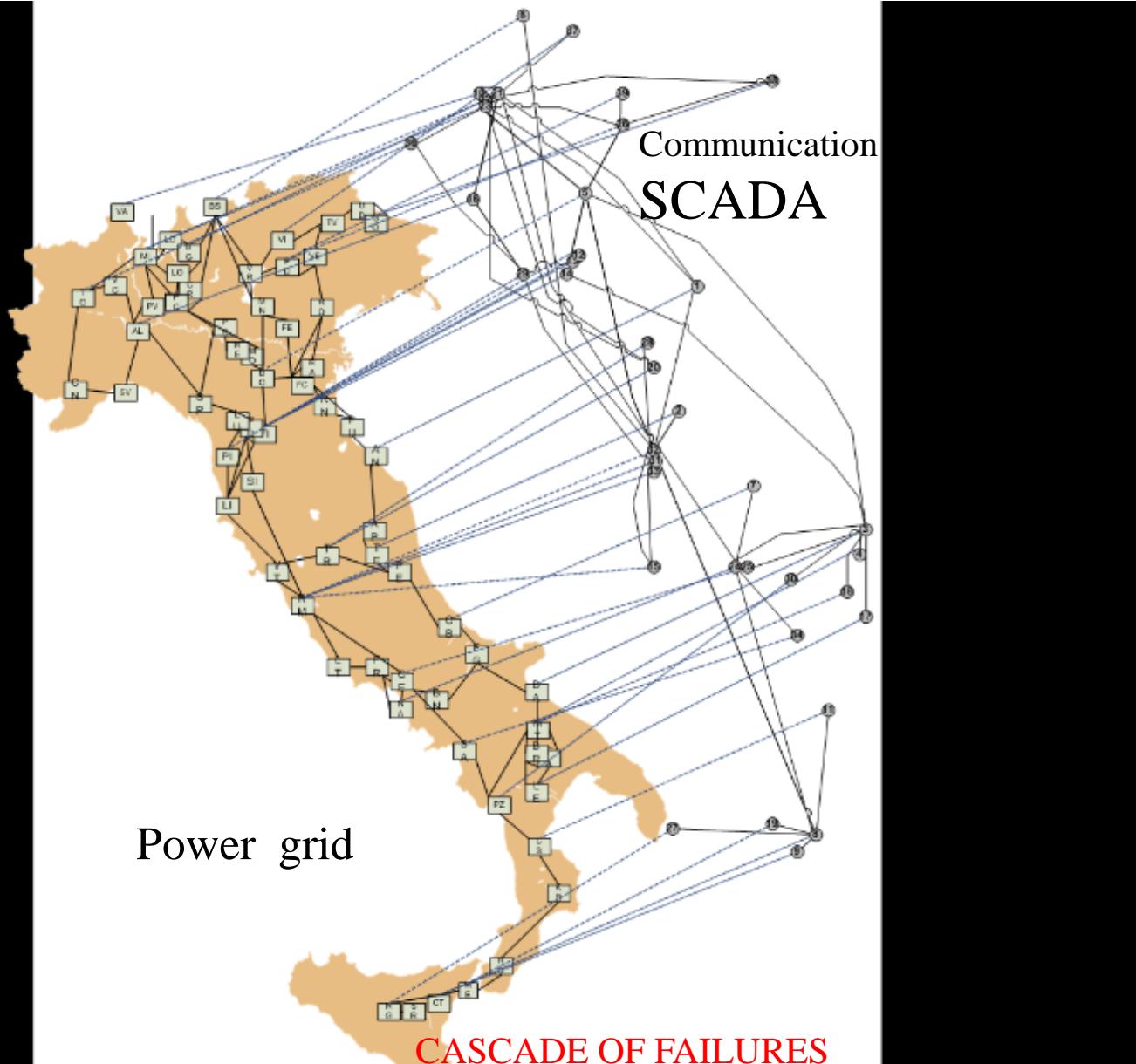
- Until 2010 studies focused on a **single network** which is isolated AND does not interact or influenced by other systems.
- Isolated systems **rarely** occur in nature or in technology -- analogous to **non-interacting** particles (molecules, spins).
- Results for **interacting networks** are strikingly **different** from those of single networks.



# Blackout in Italy (28 September 2003)-ABRUPT COLLAPSE

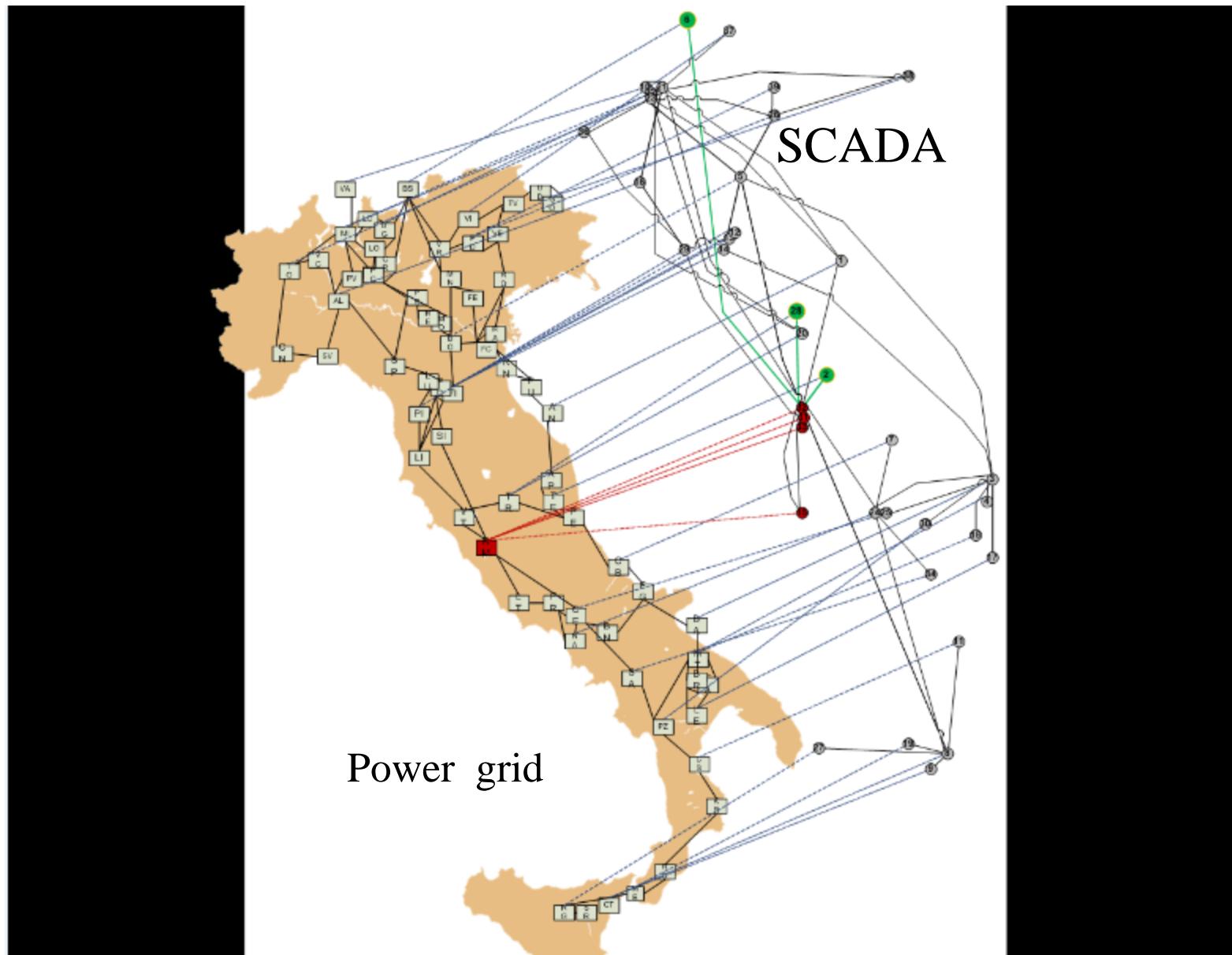
Cyber  
Attacks-  
CNN  
Simulation  
(2010)

Rosato et al  
Int. J. of Crit.  
Infrastruct. 4,  
63 (2008)



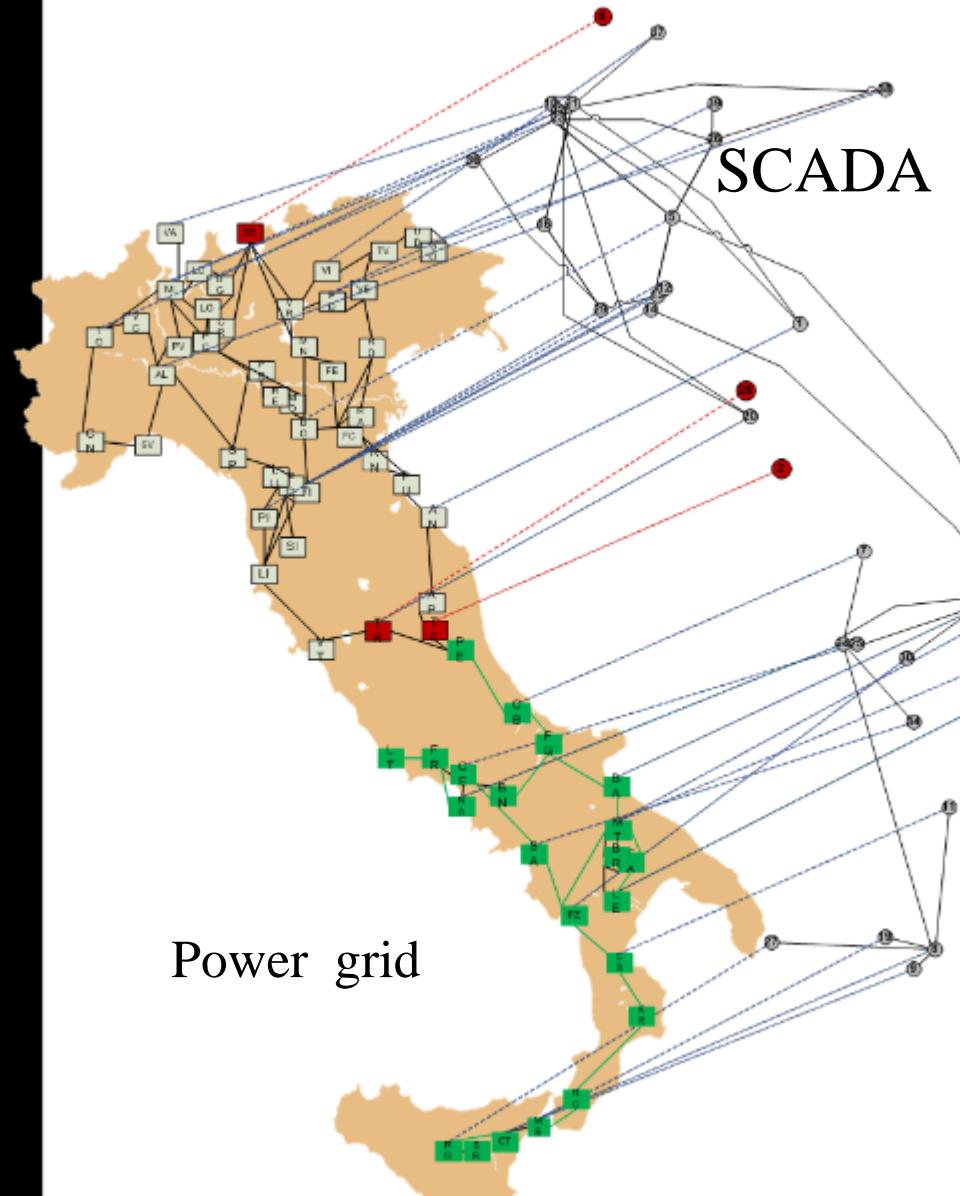
Railway network, health care systems, financial services, communication systems

# Blackout in Italy (28 September 2003)

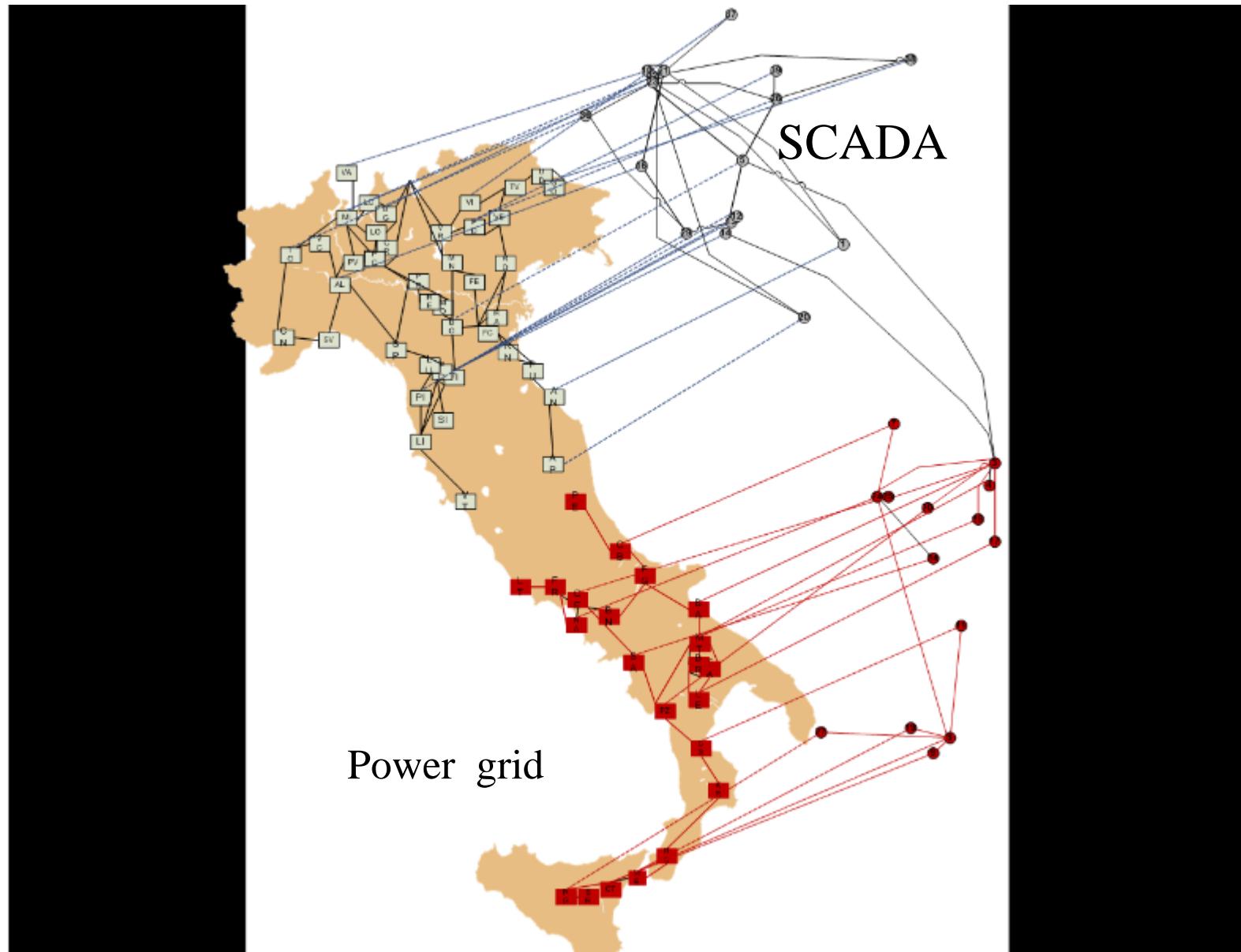


SCADA=Supervisory Control And Data Acquisition

# Blackout in Italy (28 September 2003)

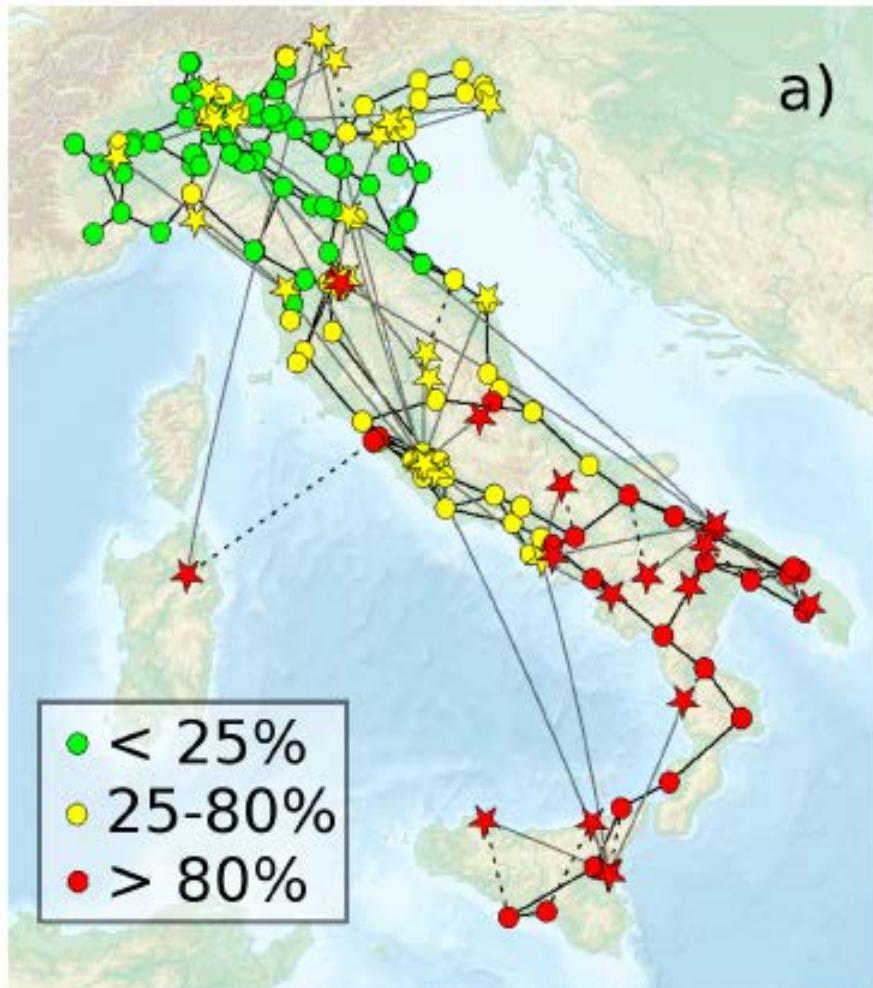


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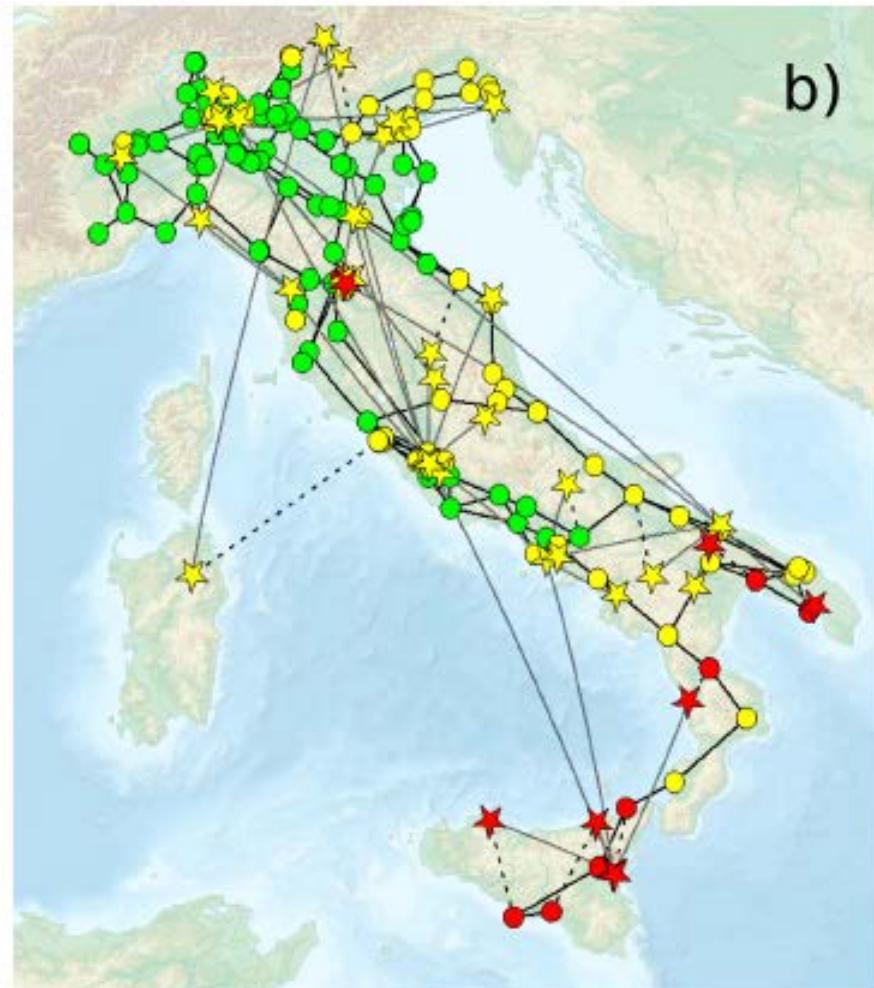


# Designing Robust Coupled Networks: Italy 2003 blackout

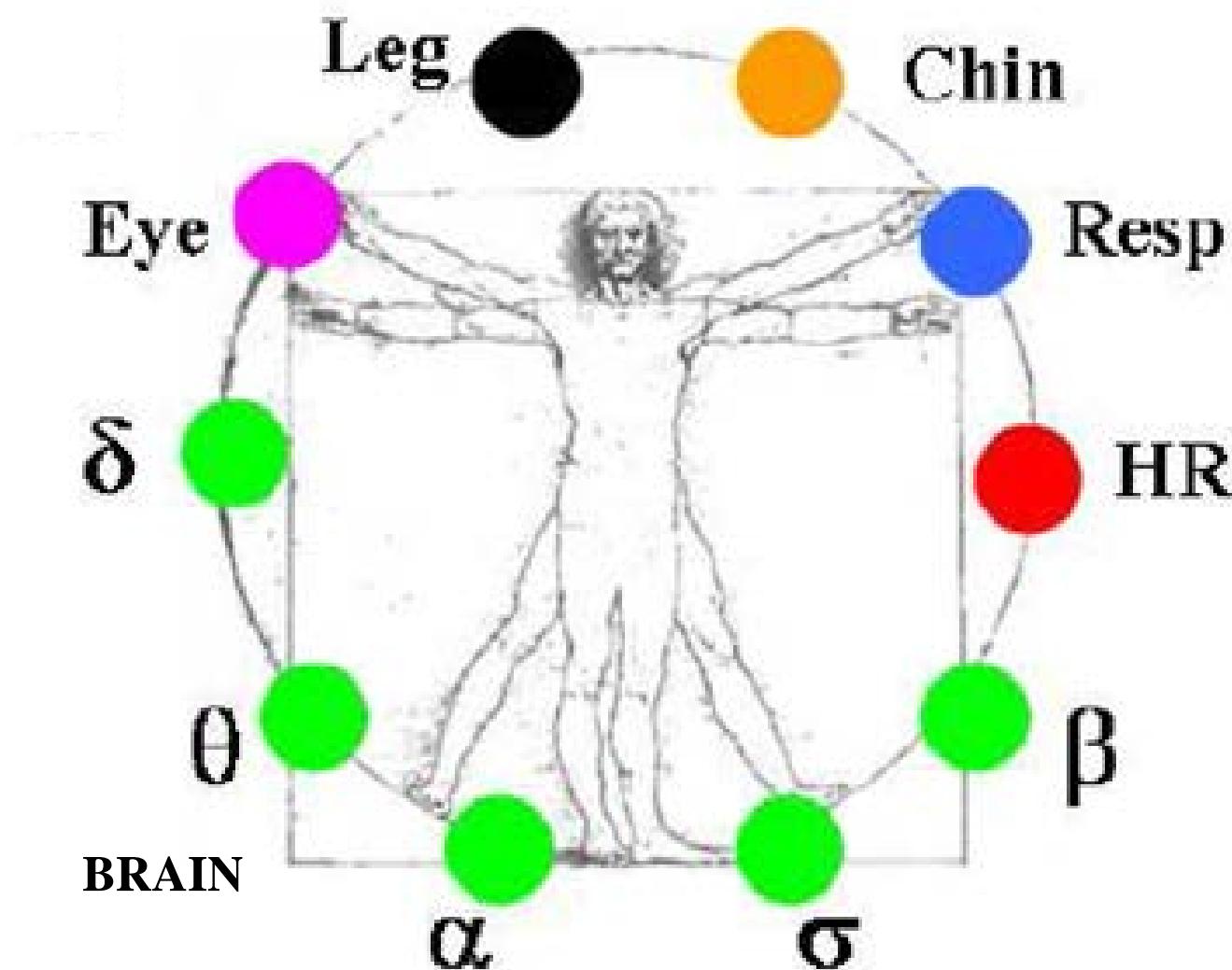
Random interdependencies



Nearly optimal interdependencies

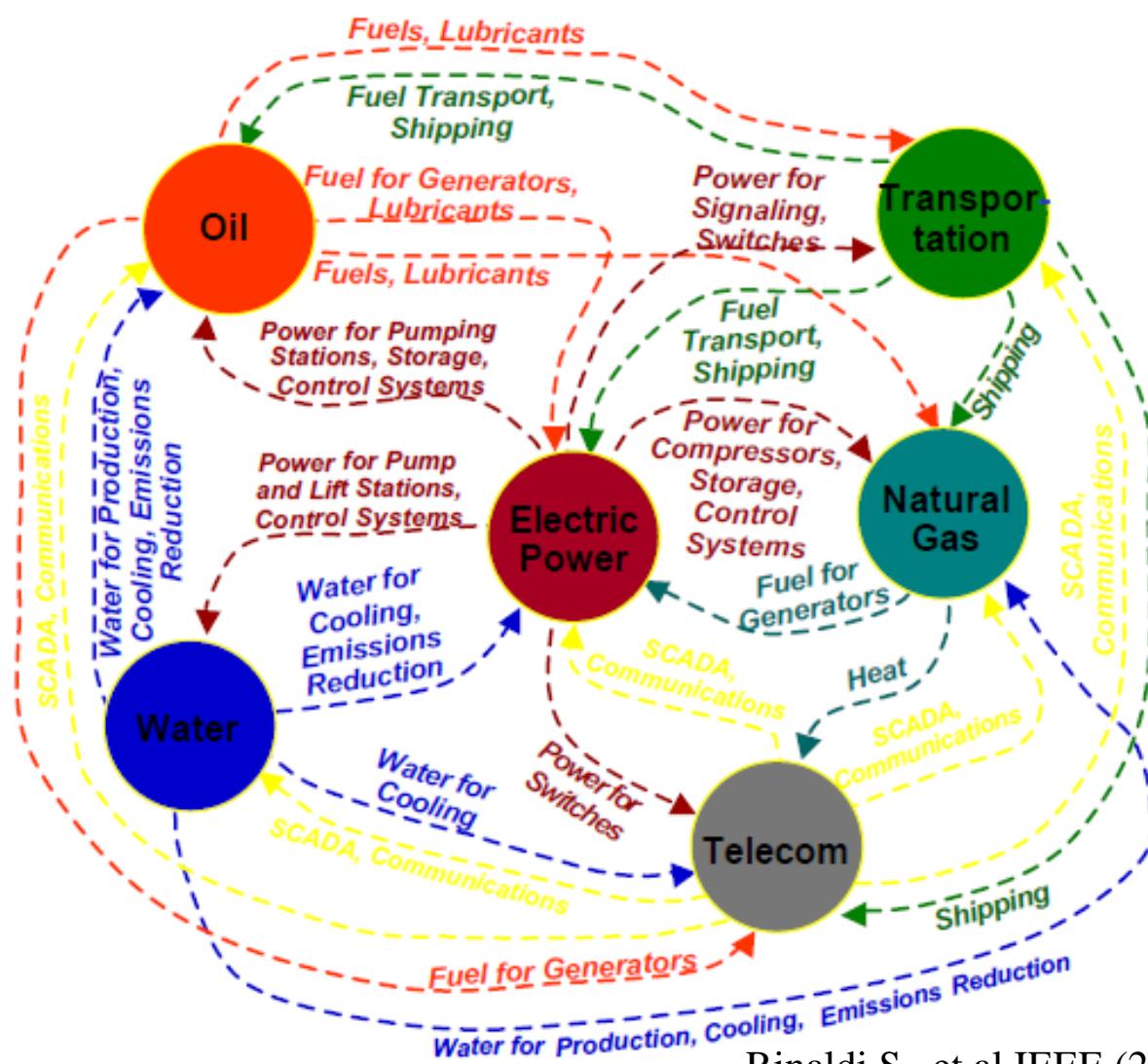


# HUMAN BODY: NETWORK OF NETWORKS



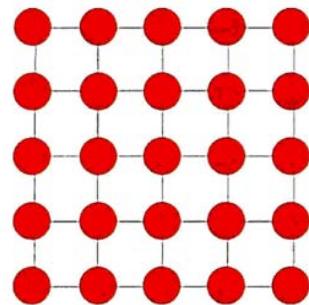
**System Collapse**

# *How interdependent are infrastructures?*



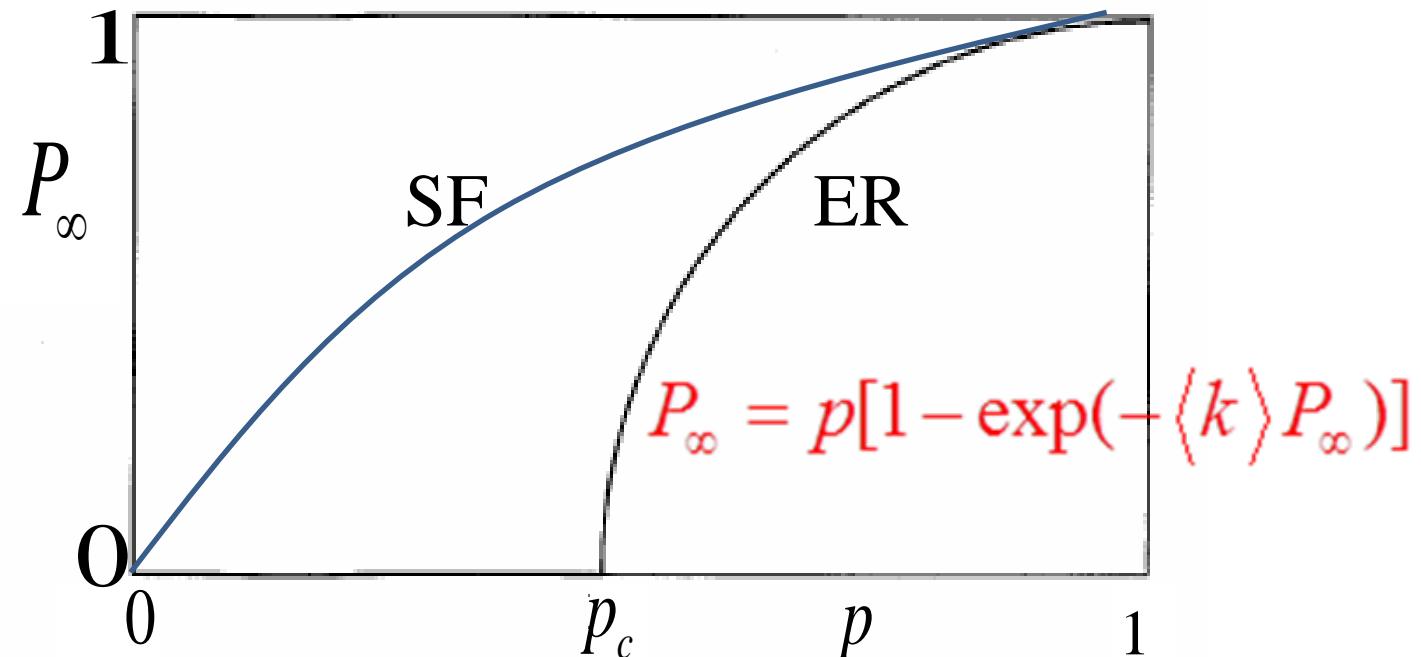
Rinaldi S, et al IEEE (2001)

# CONTINUOUS TRANSITIONS



$$p_c = 0.593$$

Long-range links  
More robust!



SF more robust!!

Classical Erdos-Renyi (1960)

Homogeneous, similar to lattices

$d \sim \log N$ -- Small world

$$p_c = 1 - q_c = 1 / \langle k \rangle$$

$$P_\infty = p[1 - \exp(-\langle k \rangle P_\infty)]$$

Barabasi-Albert (1999)

Heterogeneous-translational symmetry breaks!  
New universality class-many anomalous laws

$$\text{e.g., } d \sim \log \log N ; p_c = 0$$

Ultra Small worlds (Cohen and SH, PRL (2003))

Breakthrough in understanding many problems!

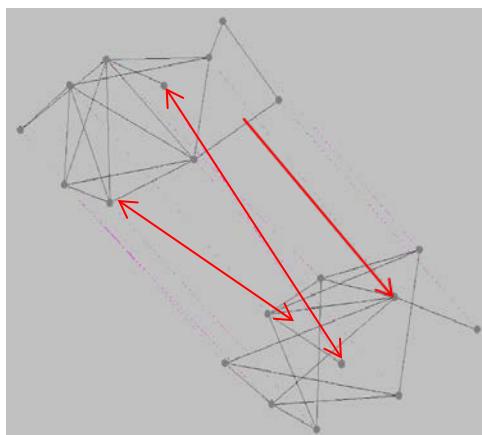
# Comparing single and coupled networks: Robustness

Remove randomly (or targeted) a fraction  $1 - p$  nodes

$P_\infty$  Size of the largest connected component (cluster)

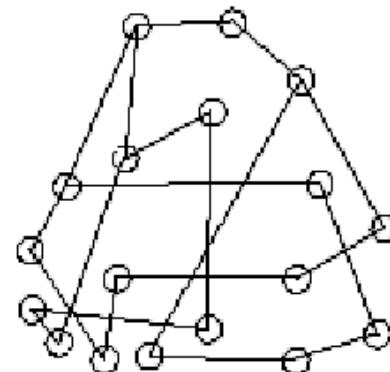
$p_c$  Breakdown threshold

**Single** networks:  
Continuous transition



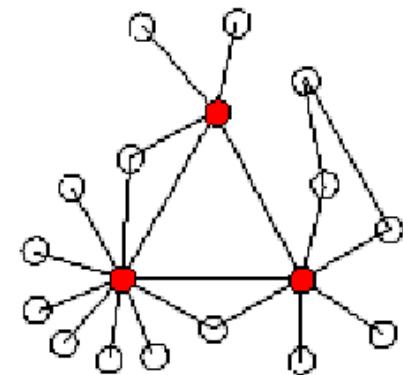
**Coupled** networks:  
New paradigm-Abrupt transition  
Cascading Failures

## Exponential (ER)

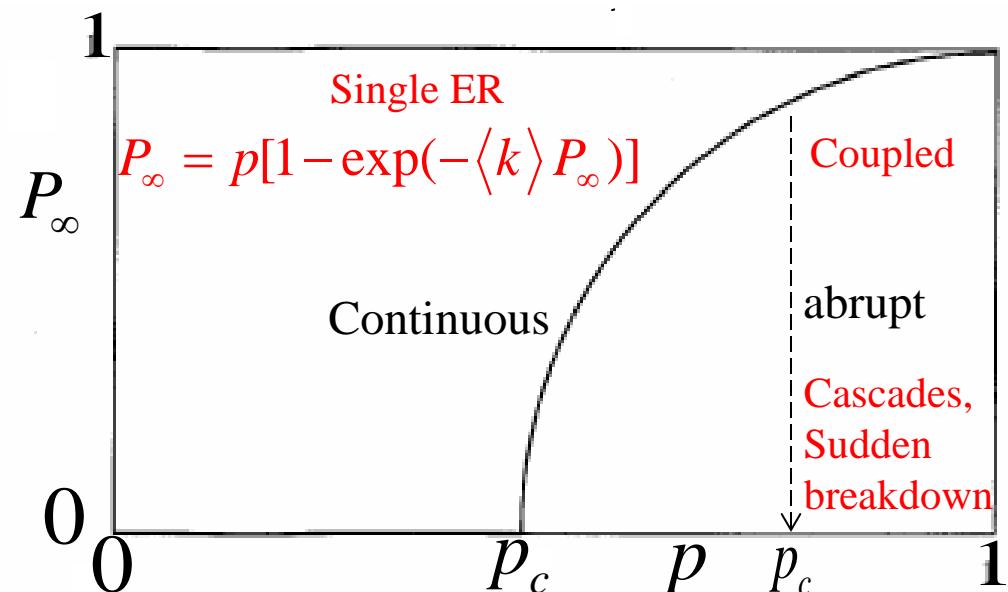


$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

## Scale-free (SF)

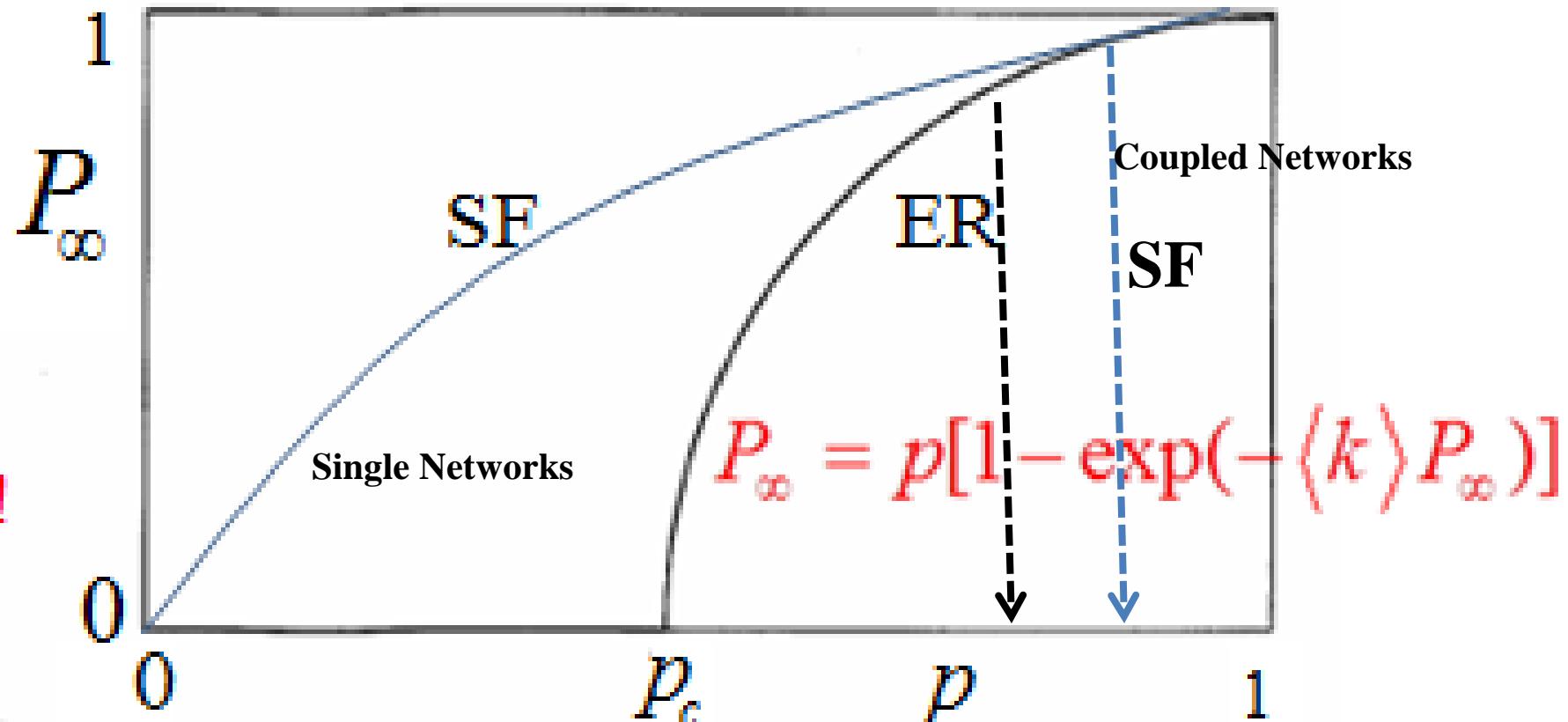


$$P(k) = \begin{cases} ck^{-\lambda} & m \leq k \leq K \\ 0 & \text{otherwise} \end{cases}$$



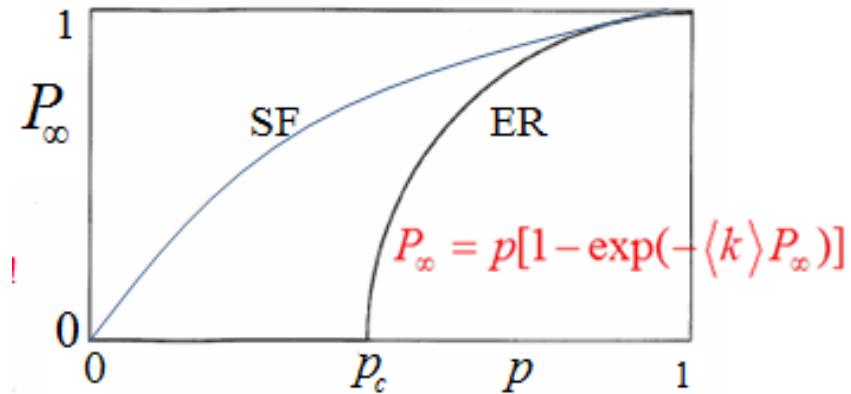
Message: our world is extremely unsafe!

IN CONTRAST TO SINGLE NETWORKS, COUPLED NETWORKS  
ARE MORE VULNERABLE WHEN DEGREE DIST. IS BROADER

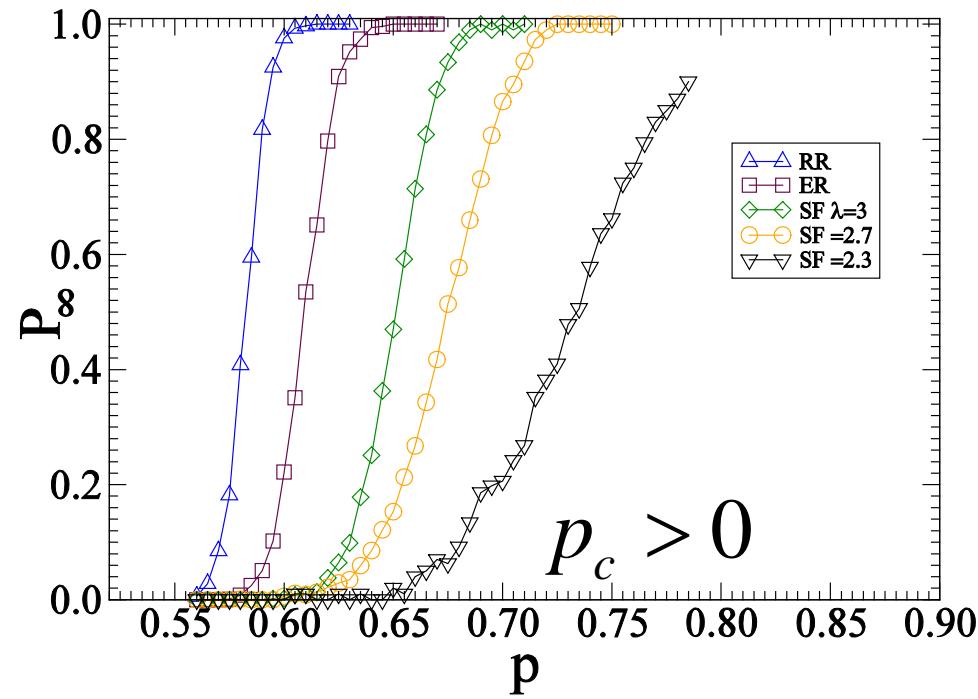


Same average degree

IN CONTRAST TO SINGLE NETWORKS, COUPLED NETWORKS ARE MORE VULNERABLE WHEN DEGREE DIST. IS BROADER



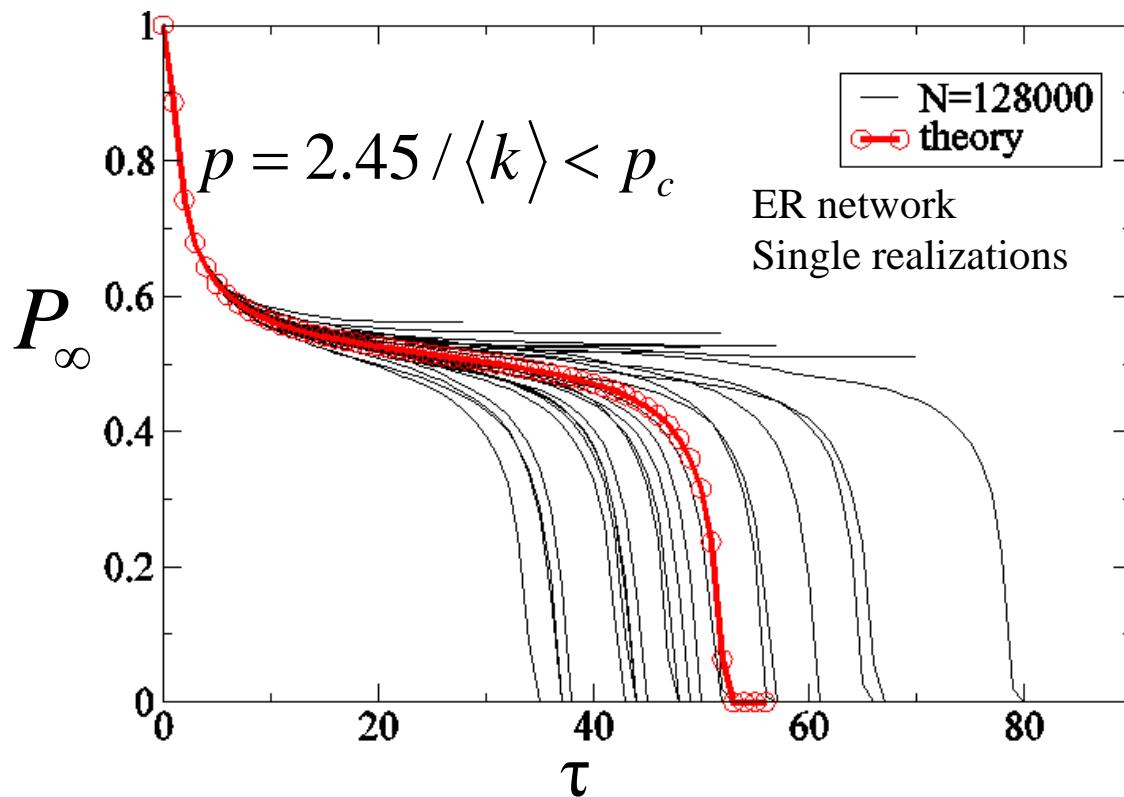
SF more robust



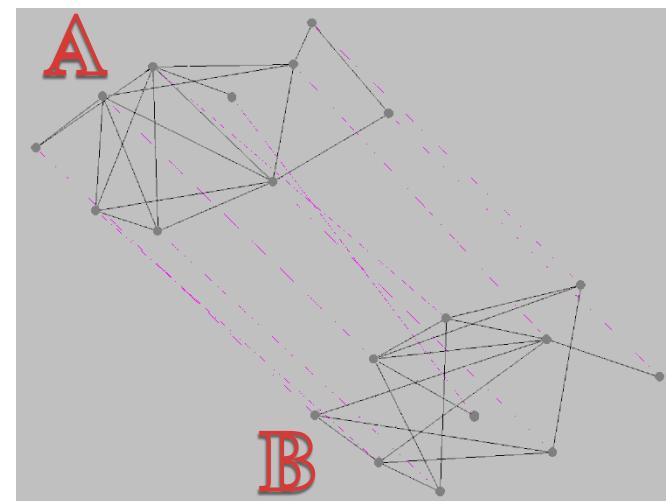
All with  $\langle k \rangle = 4$

# RESULTS: THEORY and SIMULATIONS: ER Networks

$P_\infty$  after  $\tau$ -cascades of failures



Removing 1-p nodes in A



Catastrophic cascades  
just below  $p_c$

ABRUPT TRANSITION (1<sup>st</sup> order)

$$p_c = 2.4554 / \langle k \rangle \quad \text{For a single network } p_c = 1 / \langle k \rangle$$

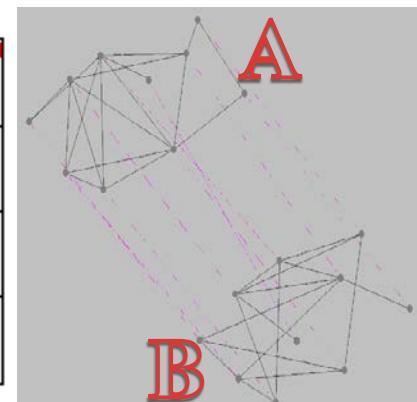
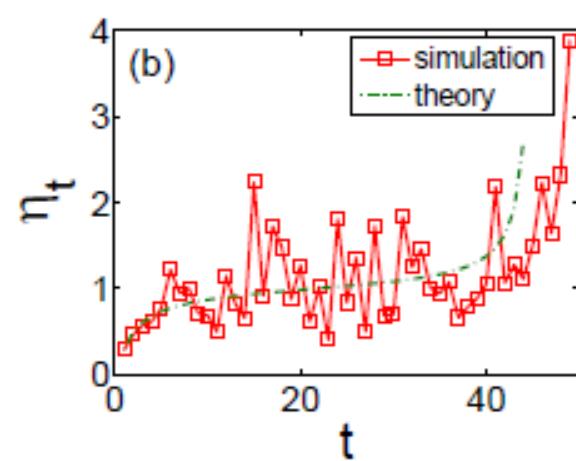
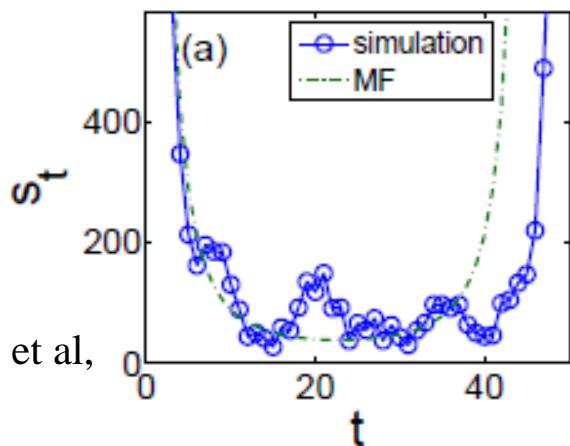
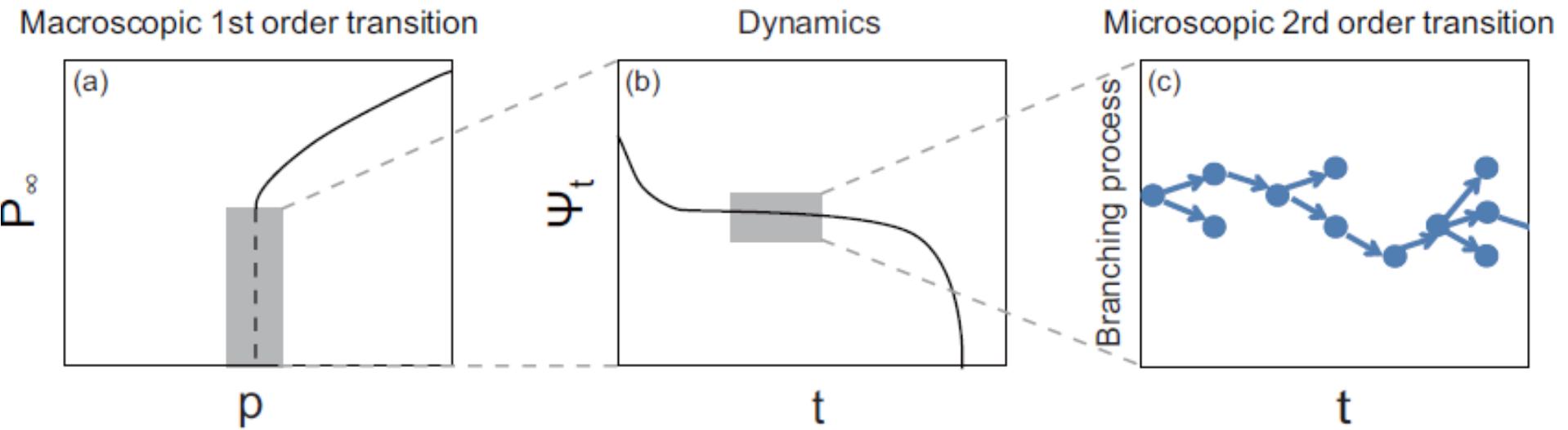
$$\langle k \rangle_{\min} = 2.4554 \text{ for single network } \langle k \rangle_{\min} = 1$$

$$\langle \tau \rangle \sim N^{1/3}$$

Dong Zhou et al (2013)

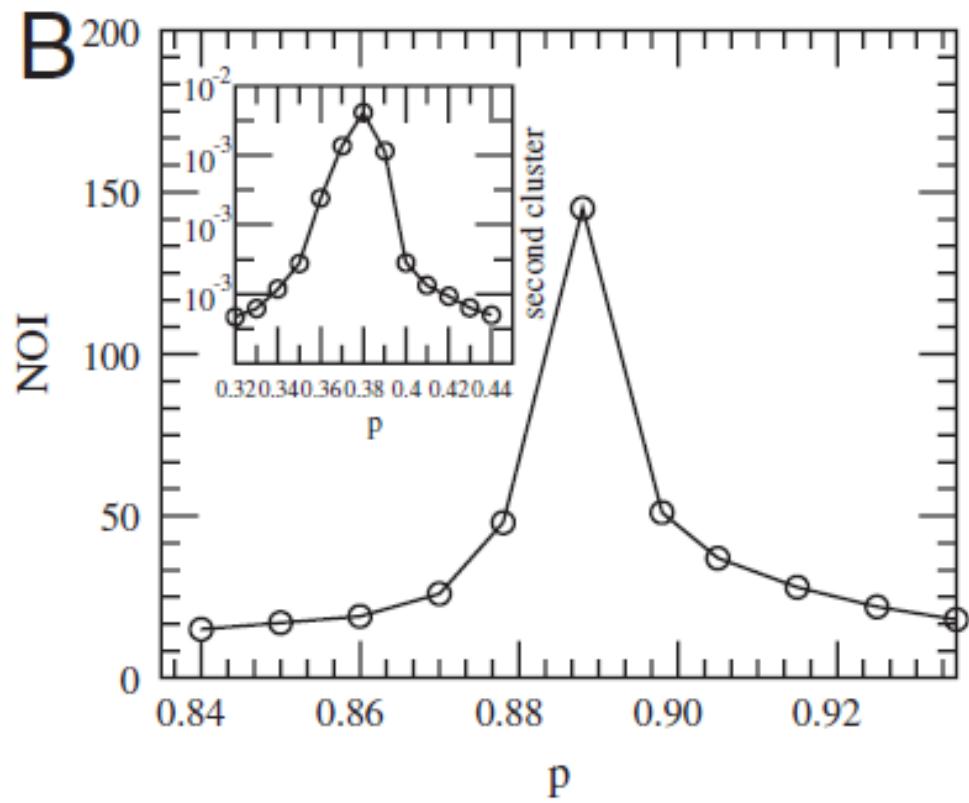
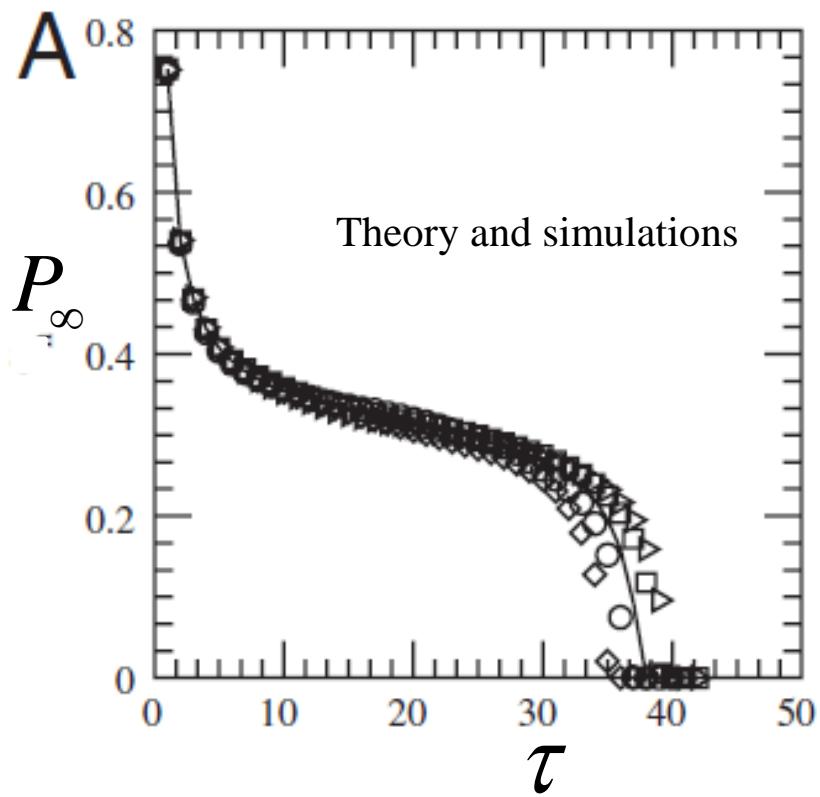
# Origin of Plateau

Simultaneous first and second order percolation transitions

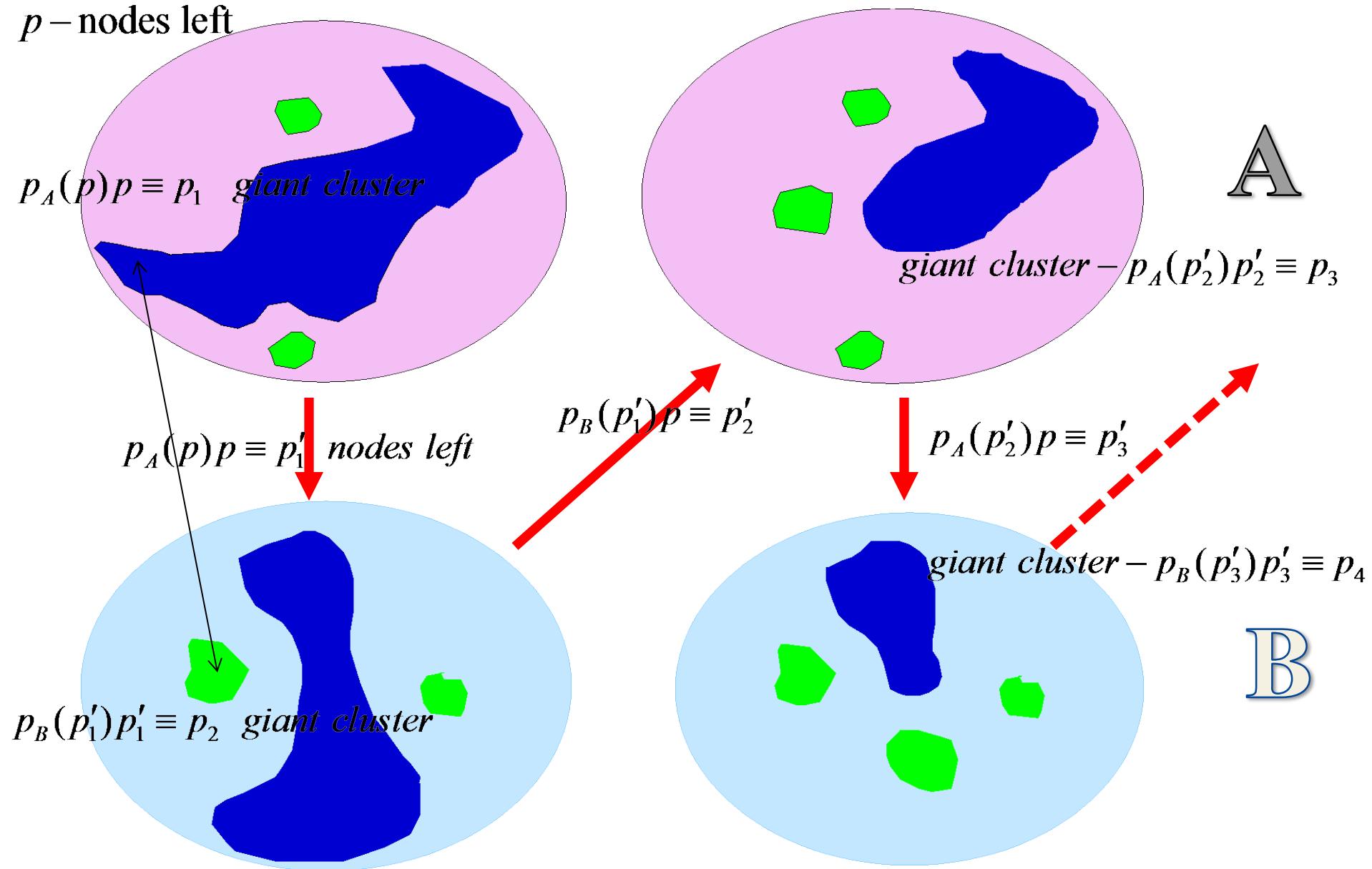


# Interdependent Networks

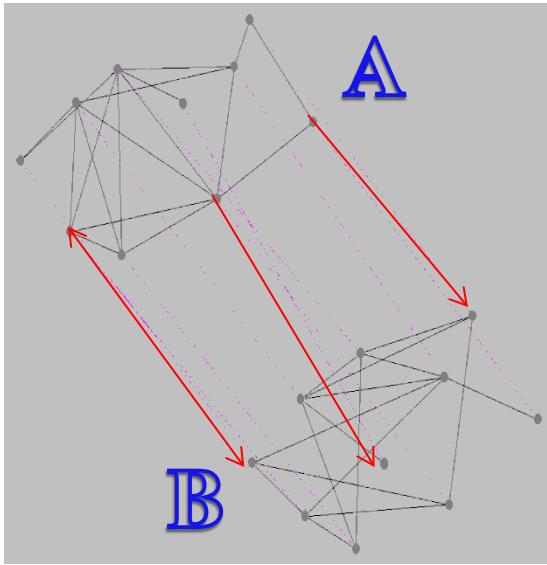
Determining  $p_c$  in simulations:



# RANDOM REMOVAL – PERCOLATION FRAMEWORK



# GENERALIZATION: PARTIAL DEPENDENCE: Theory and Simulations

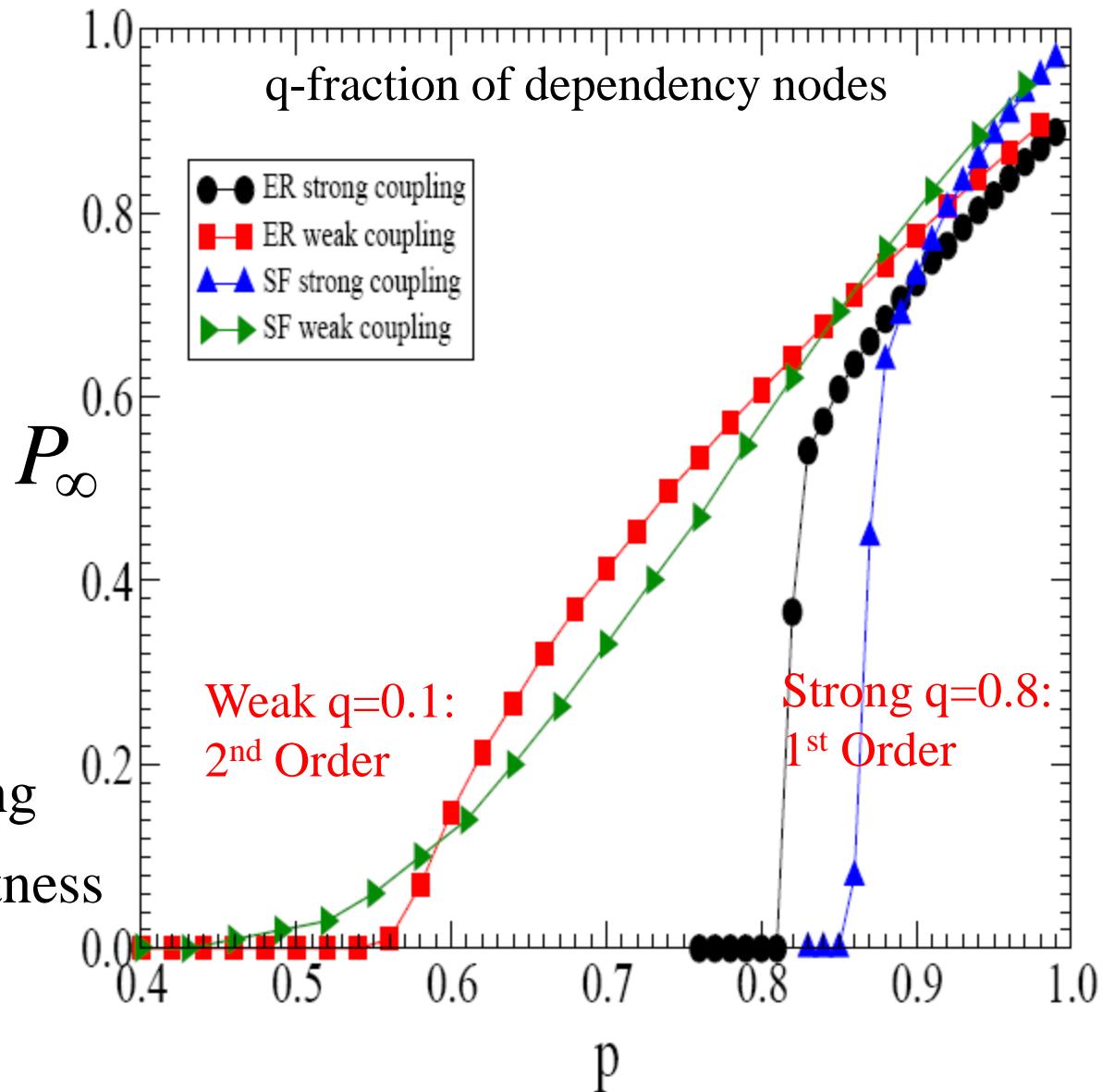


Parshani, Buldyrev, S.H.  
PRL, 105, 048701 (2010)

$q_c \approx 0.2$  for random coupling

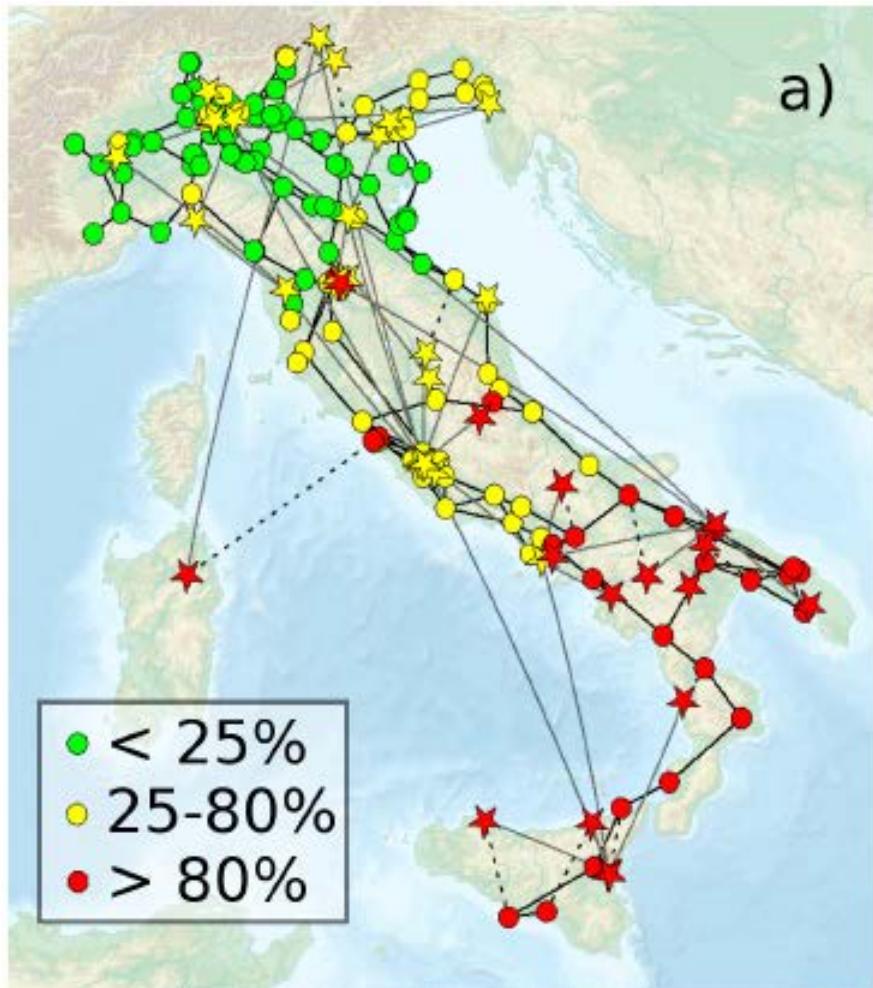
$q_c \rightarrow 0.9$  for optimal robustness

Schneider et al [arXiv:1106.3234](https://arxiv.org/abs/1106.3234)  
Scientific Reports (2013)

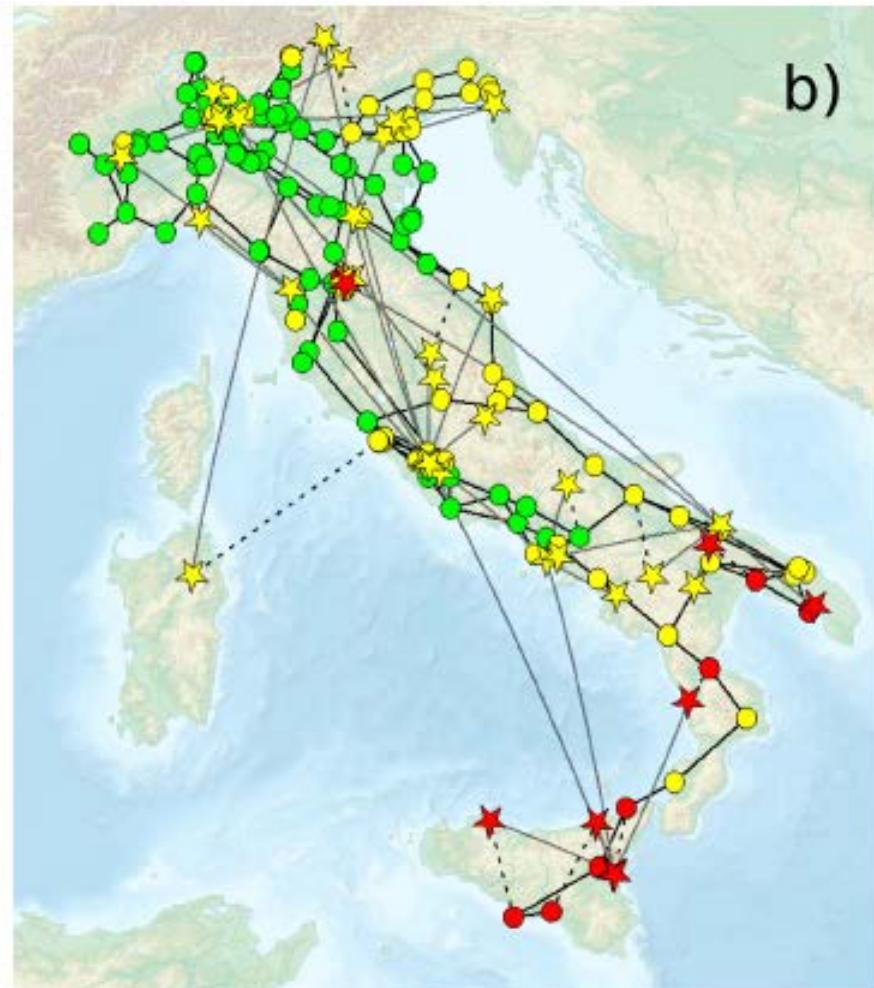


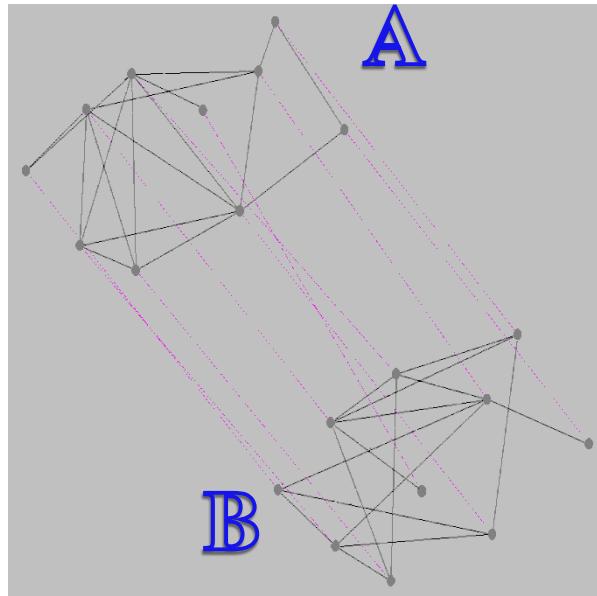
# Designing Robust Coupled Networks: Italy 2003 blackout

Random interdependencies

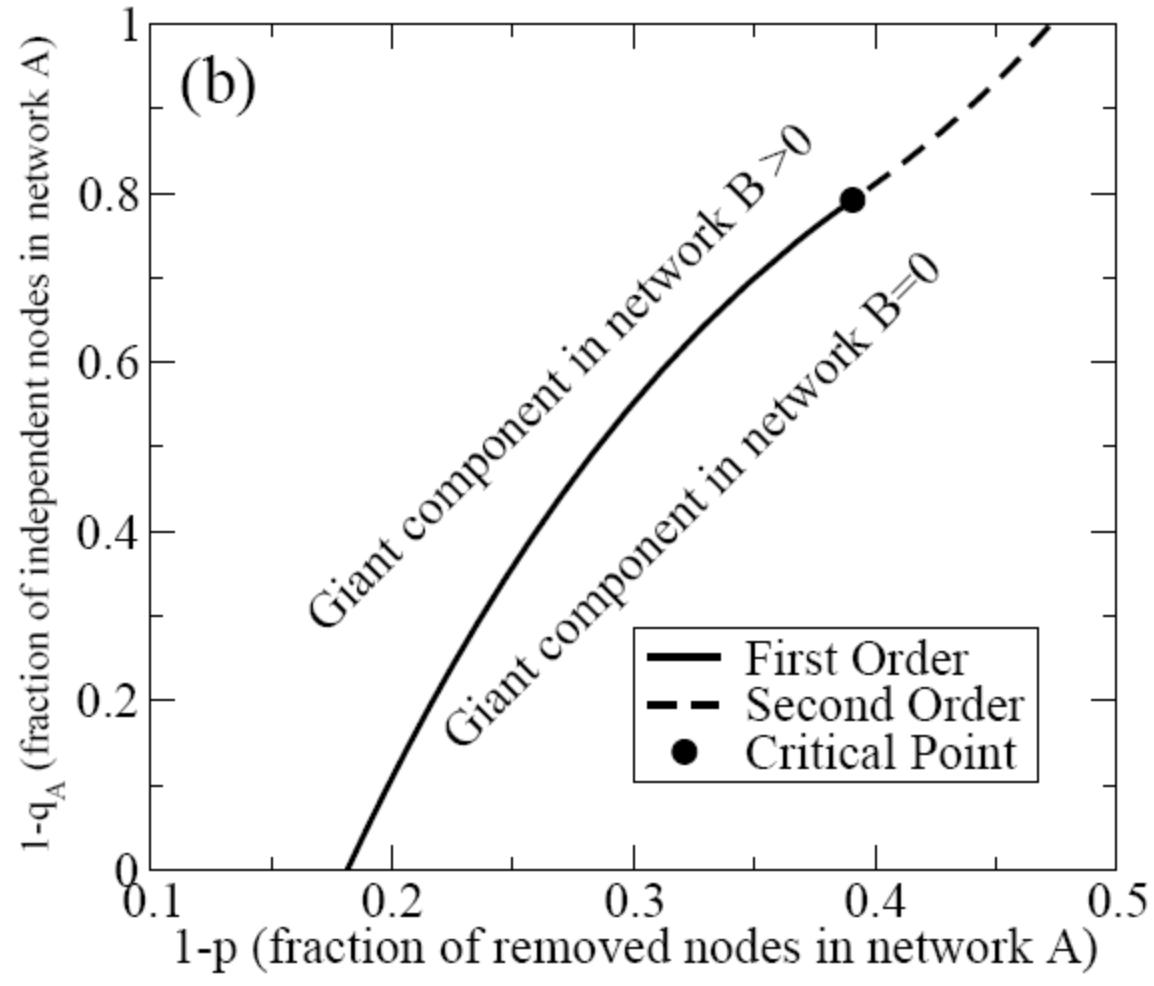
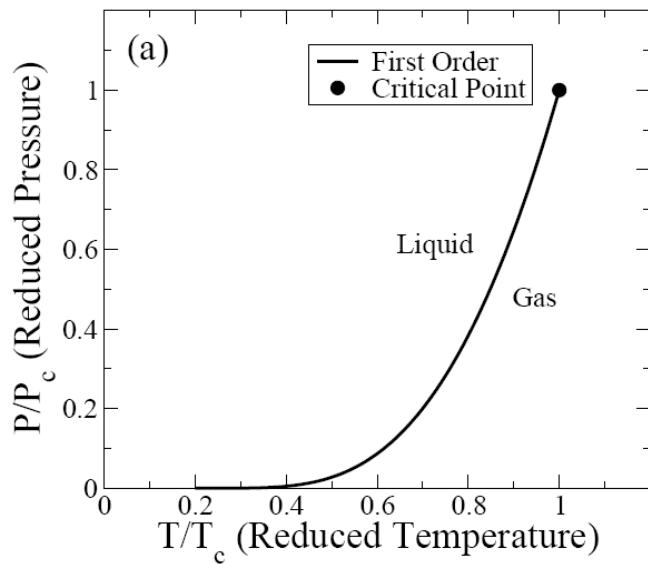


Nearly optimal interdependencies





Analogous to **critical point**  
in liquid-gas transition:

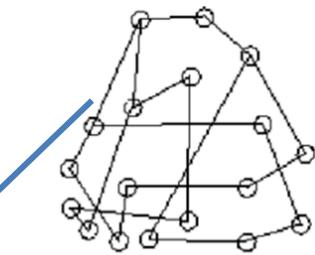
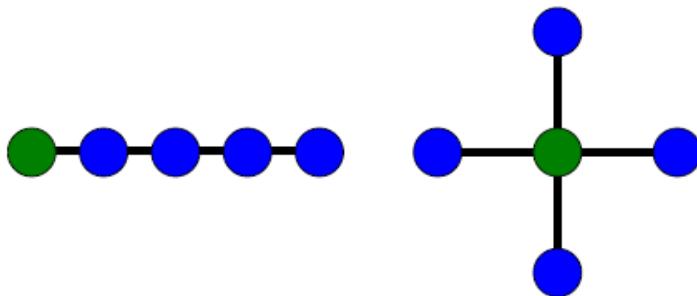


# PARTIAL DEPENDENCE: Tricritical point

Parshani et al  
PRL, **105**, 048701 (2010)

# Network of Networks (tree)

$n=5$



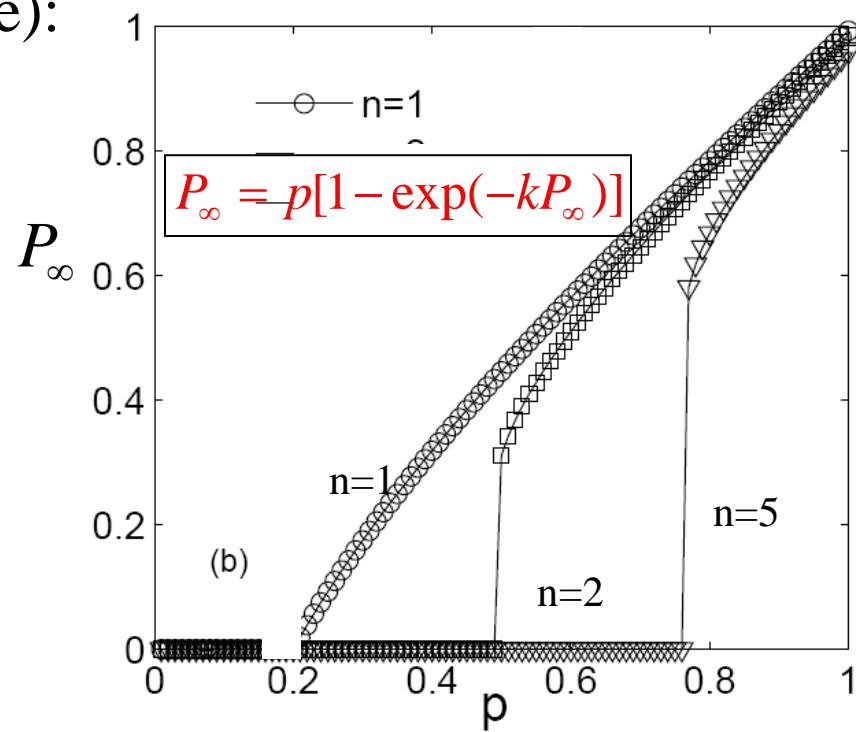
For ER,  $\langle k_i \rangle = k$ , full coupling,  
ALL loopless topologies (chain, star, tree):

$$P_\infty = p[1 - \exp(-kP_\infty)]^n$$

$n=1$  known ER- 2<sup>nd</sup> order

$$p_c = 1 / \langle k \rangle$$

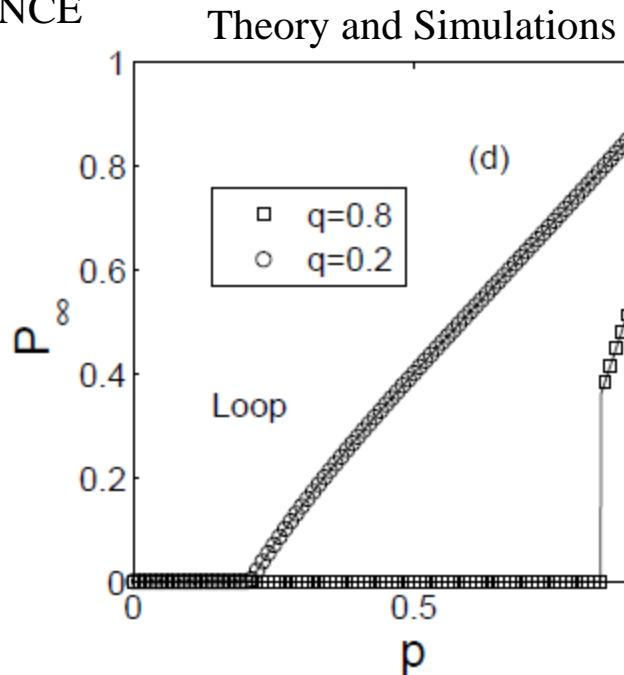
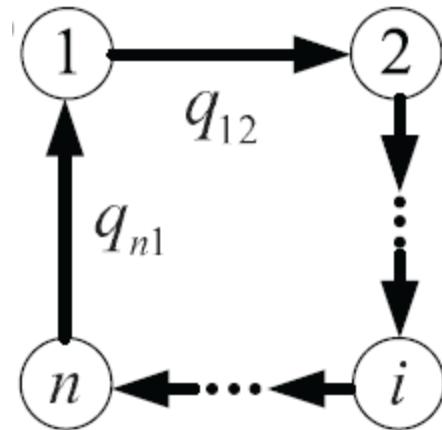
Vulnerability increases significantly with  $n$



Gao et al PRL (2011)

# Network of Networks (loop)

GENERAL FRAMEWORK -- PARTIAL DEPENDENCE



For ER networks:

$$P_\infty = p[1 - \exp(-kP_\infty)(qP_\infty - q + 1)]$$

No dependence on  $n$

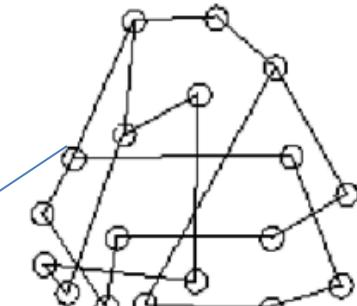
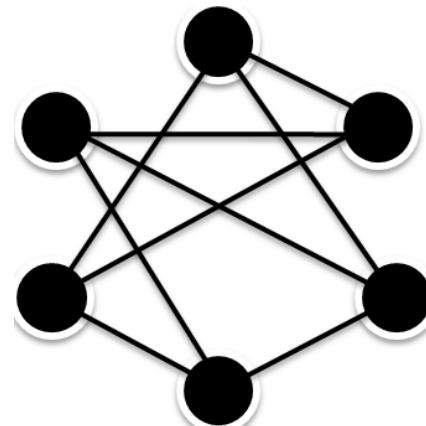
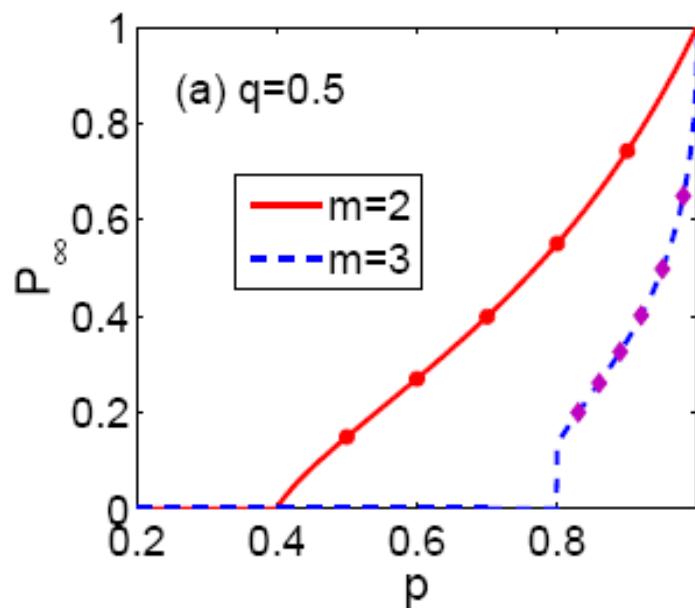
For  $q = 1$ ,  $P_\infty = 0$  -no giant component for any  $p$

For  $q = 0$ , the known single network result

$$P_\infty = p[1 - \exp(-kP_\infty)]$$

Jianxi Gao et al, PRL (2011)

# Random Regular Network of ER networks



ER  $\langle k \rangle = 2.2$

RR,  $m=3$

$$P_\infty = \frac{p}{2^m} (1 - e^{-\langle k \rangle P_\infty}) [1 - q + \sqrt{(1 - q)^2 + 4qP_\infty}]^m$$

$$p_c = \frac{1}{\langle k \rangle (1 - q)^m}$$

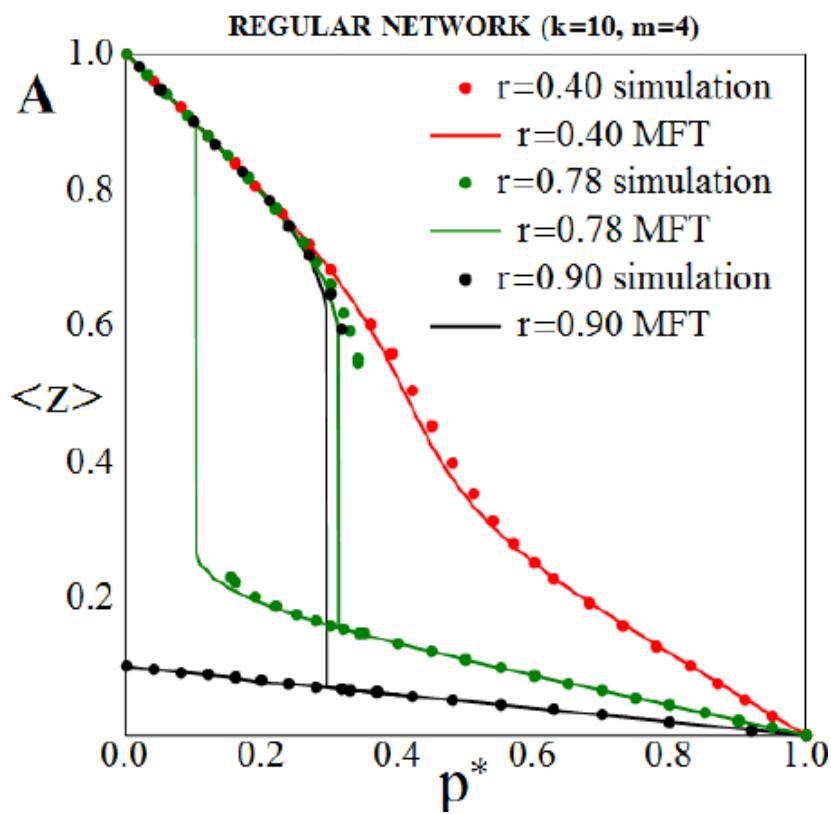
$$q_c = \frac{\langle k \rangle + m - \sqrt{m^2 + 2\langle k \rangle m}}{\langle k \rangle}$$

Surprisingly Independent on n!

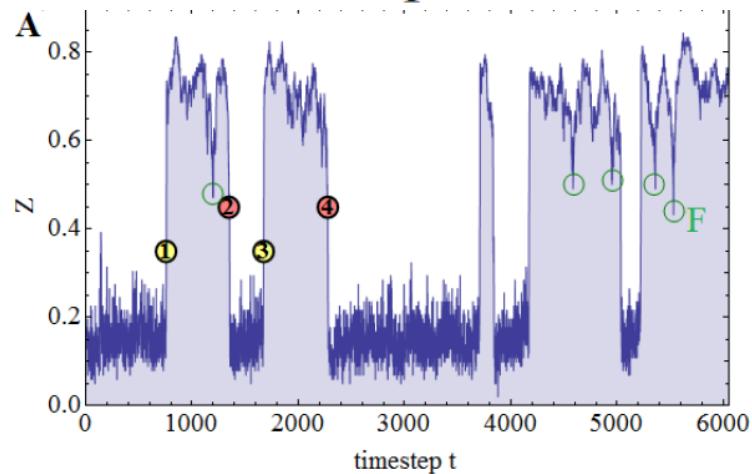
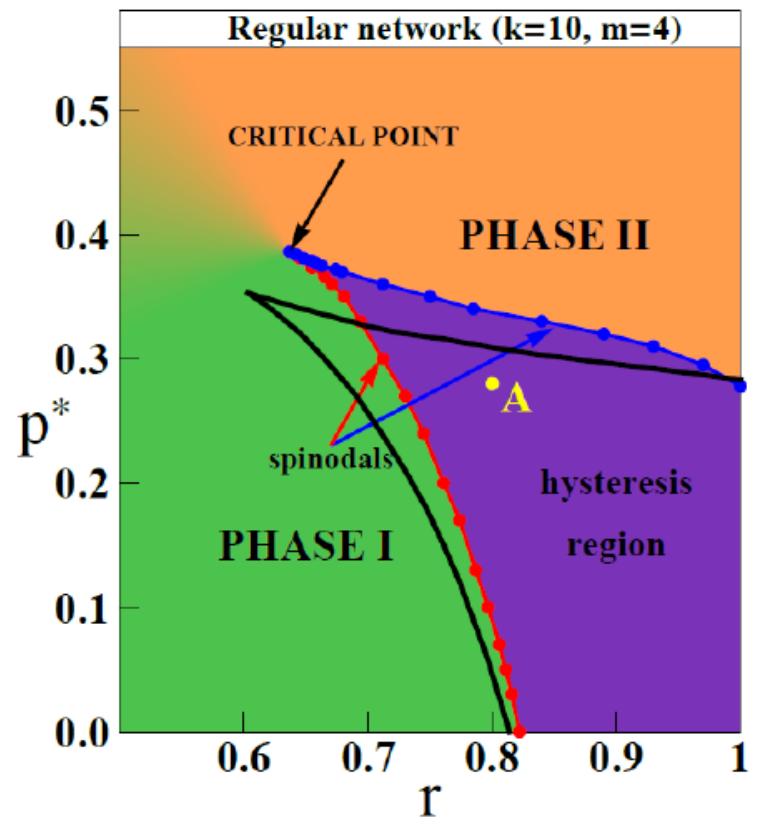
For  $m = 0$  OR  $q = 0$   
the single network  
is obtained!

$$P_\infty = p[1 - \exp(-\langle k \rangle P_\infty)]$$

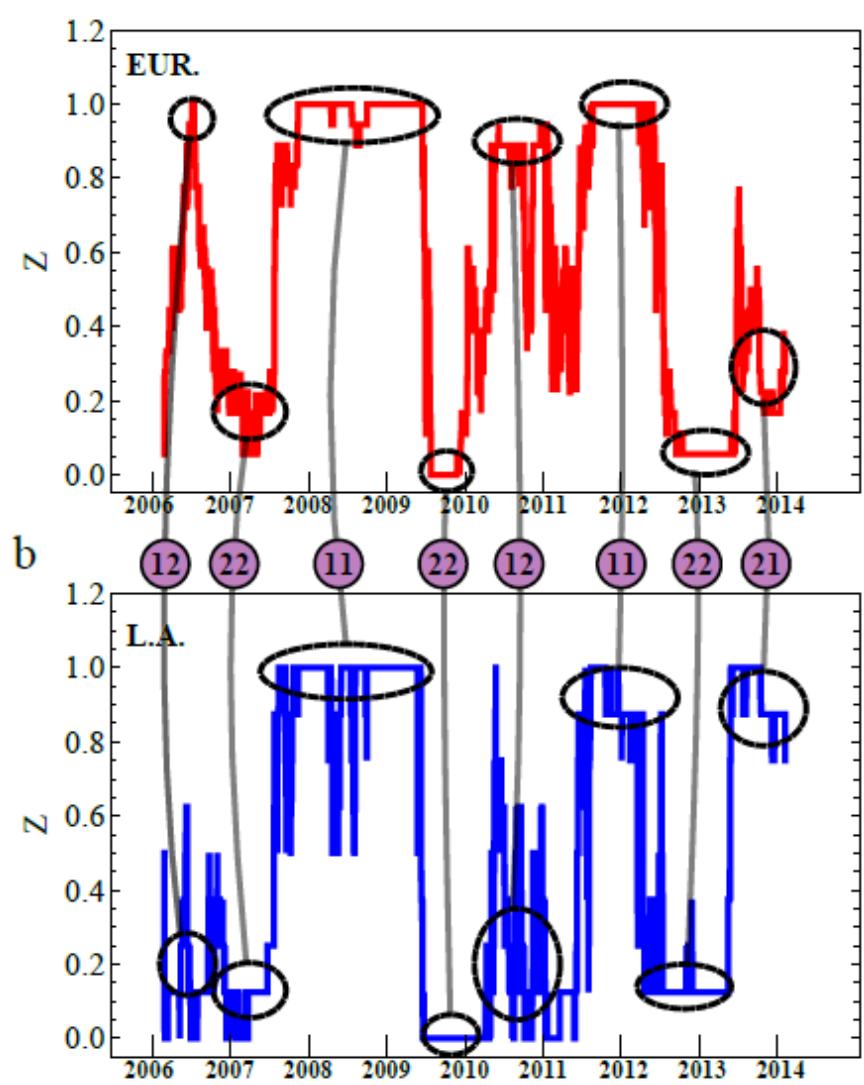
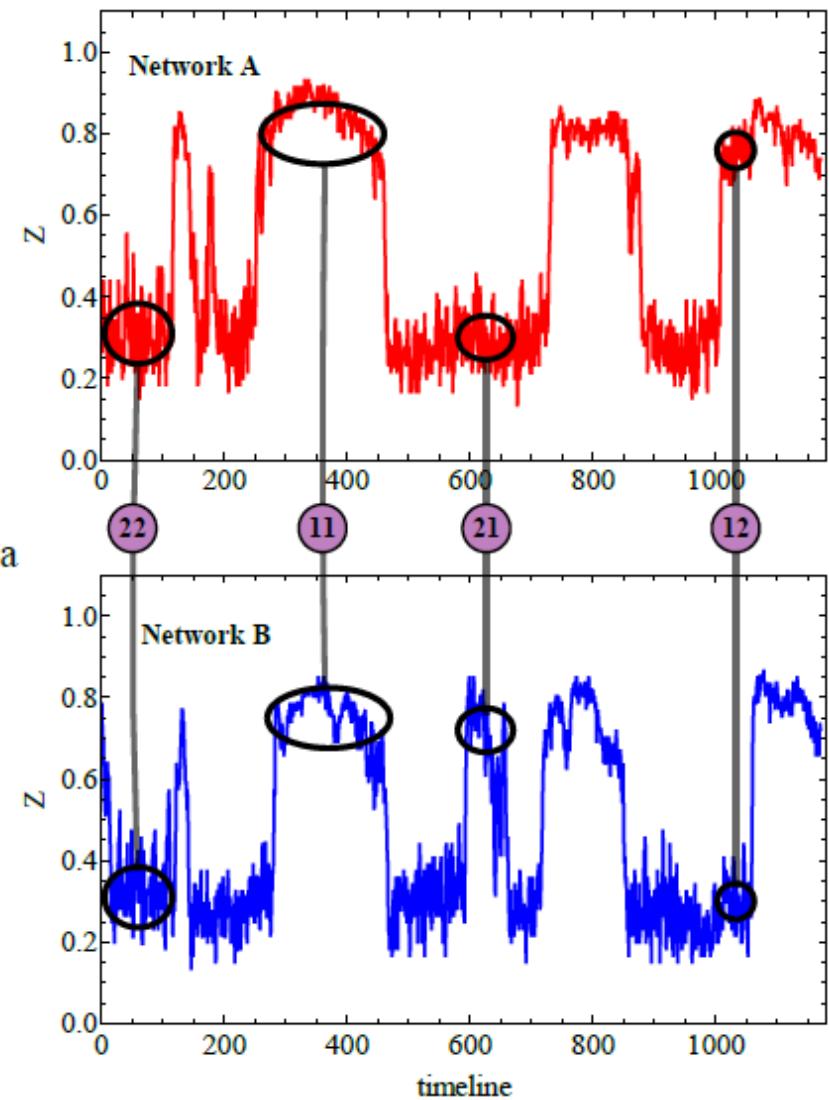
# Introducing Recovery-Single Networks



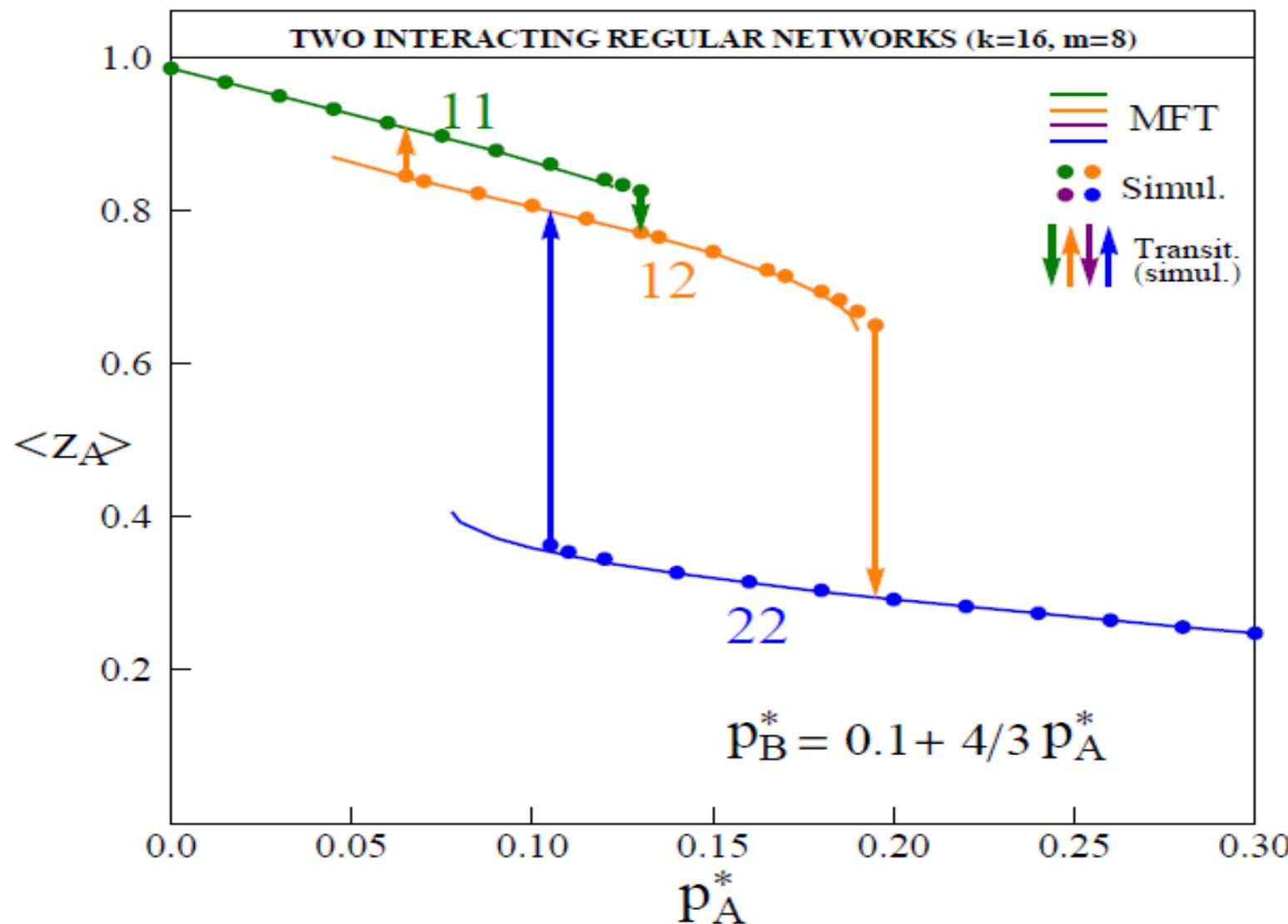
Spontaneous Recovery



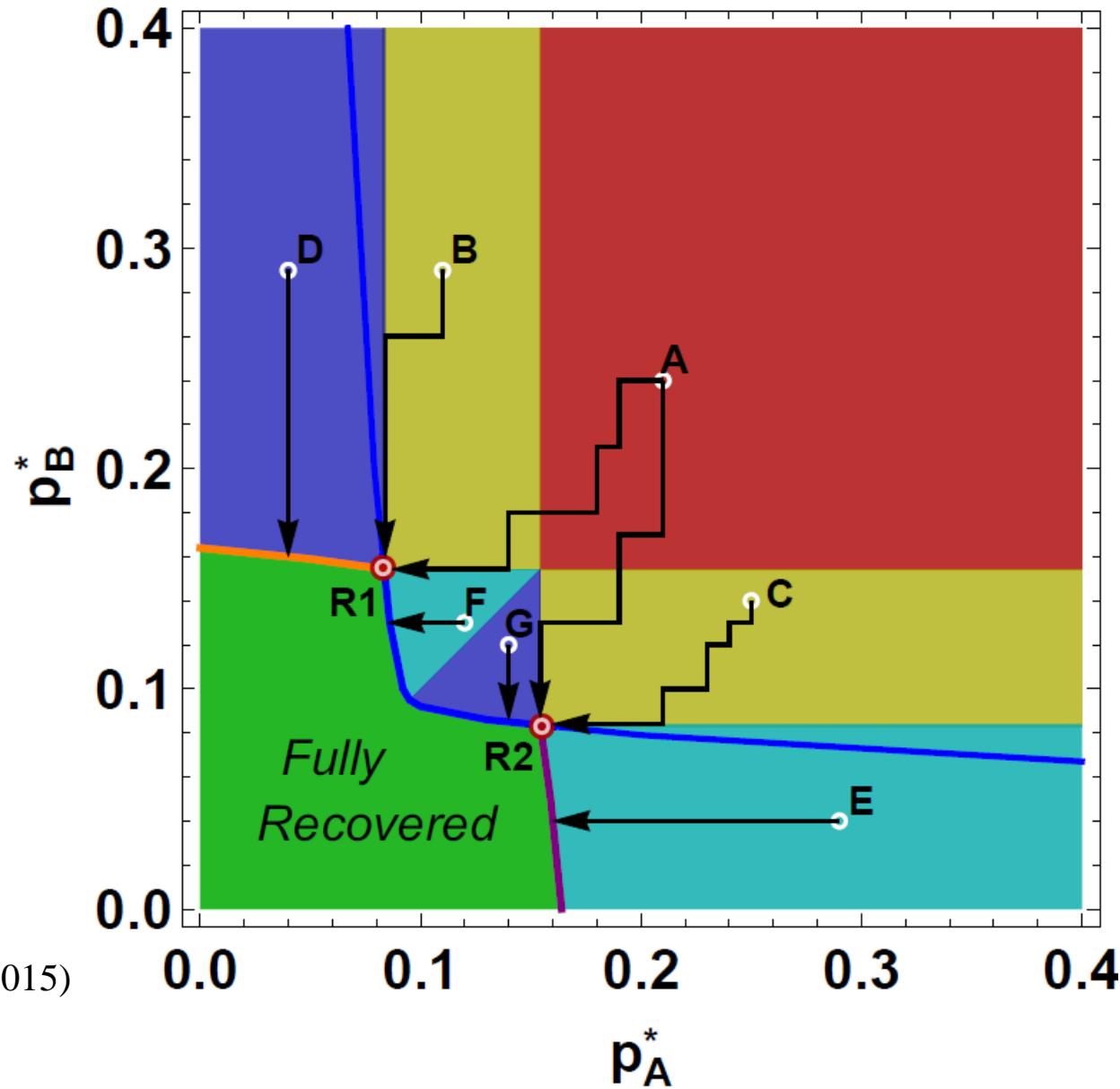
# Simultaneous Recovery and Failure of Interdependent Networks



# Complex Hysteresis-Interdependent Networks

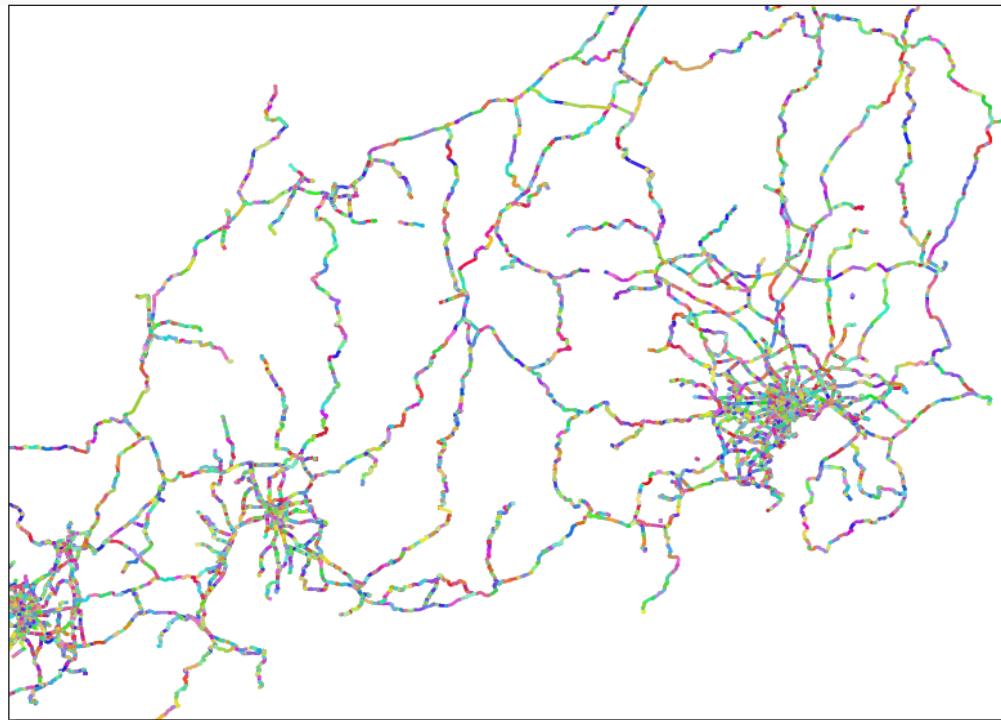


# Optimal repairing strategies



# HOW DOES SPATIALITY AFFECT TOPOLOGY?

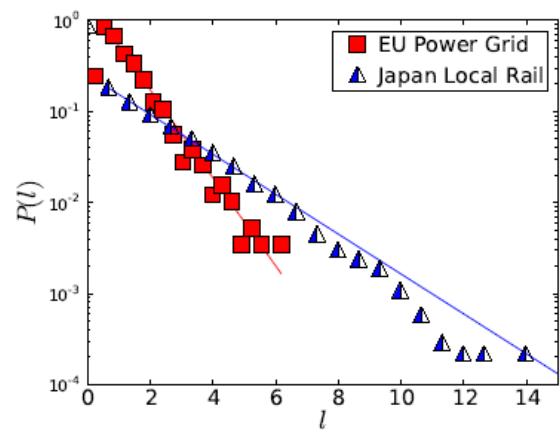
Some networks are more spatial.



Japan Railway Network



Exponential-characteristic length



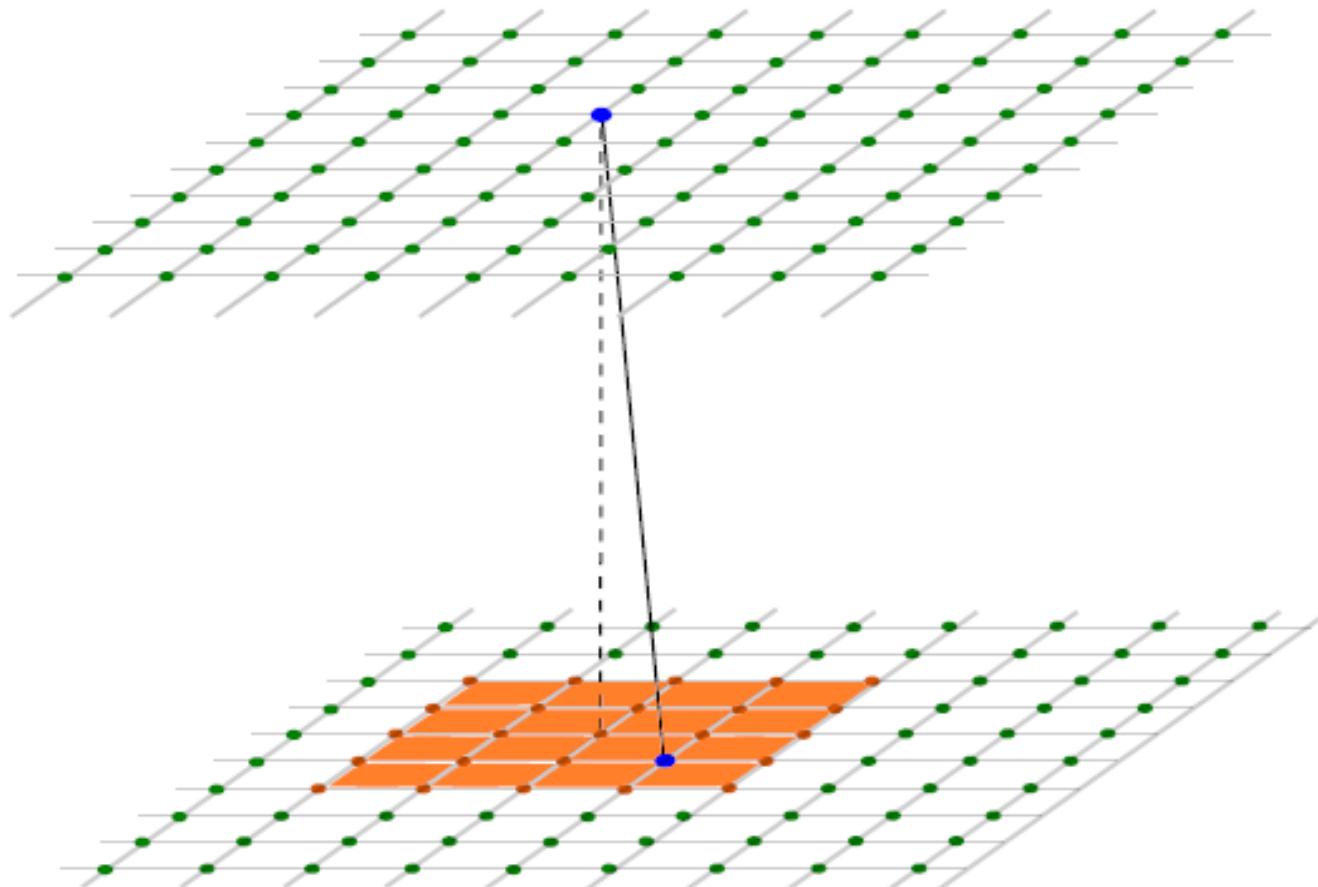
# HOW DOES SPATIALITY AFFECT TOPOLOGY?

And some are less spatial.



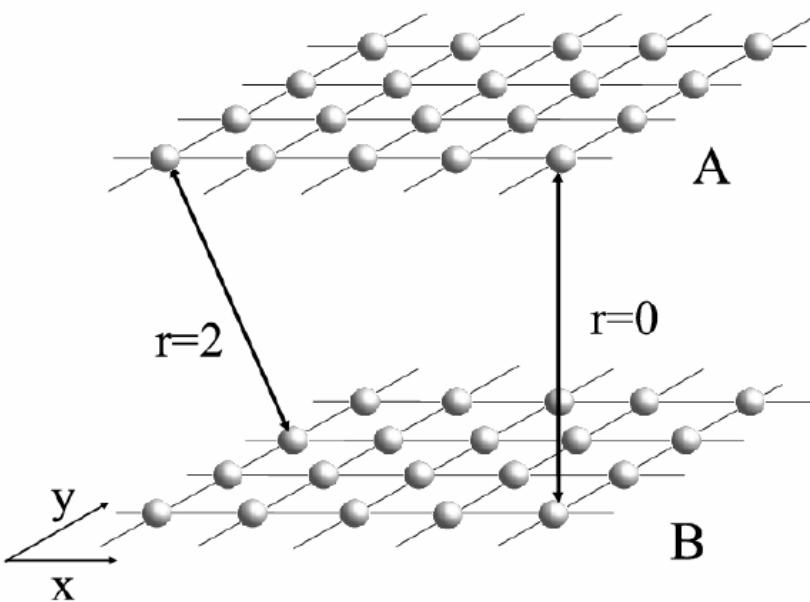
Global airline network-no characteristic length

# Embedded interdependent networks

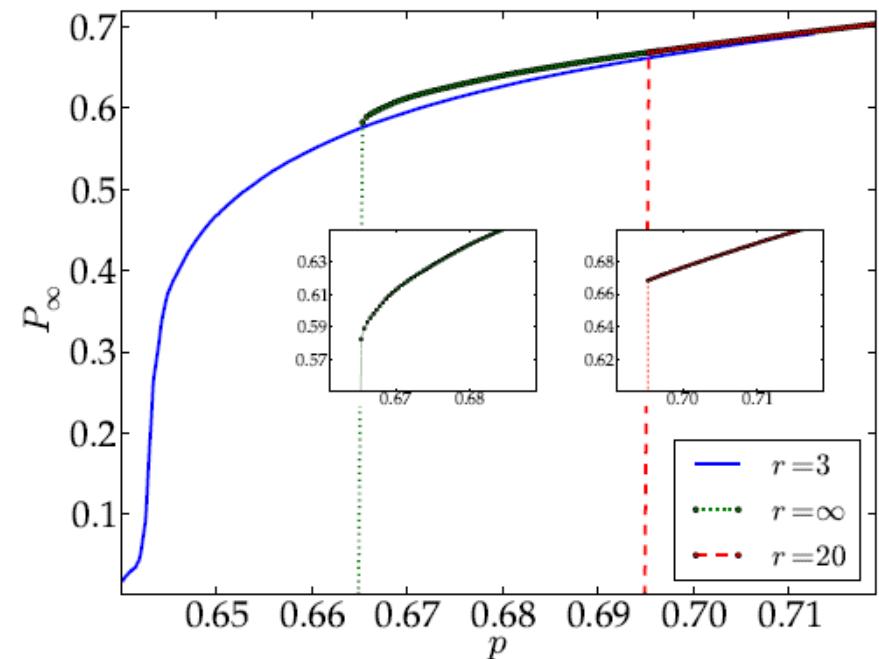


- Connectivity links: underlying lattice structure.
- Dependency links: finite dependency length  $r$ .

# Interdependent Spatially Embedded Networks



Many networks are spatially embedded:  
Internet, Power grid, Transportation etc

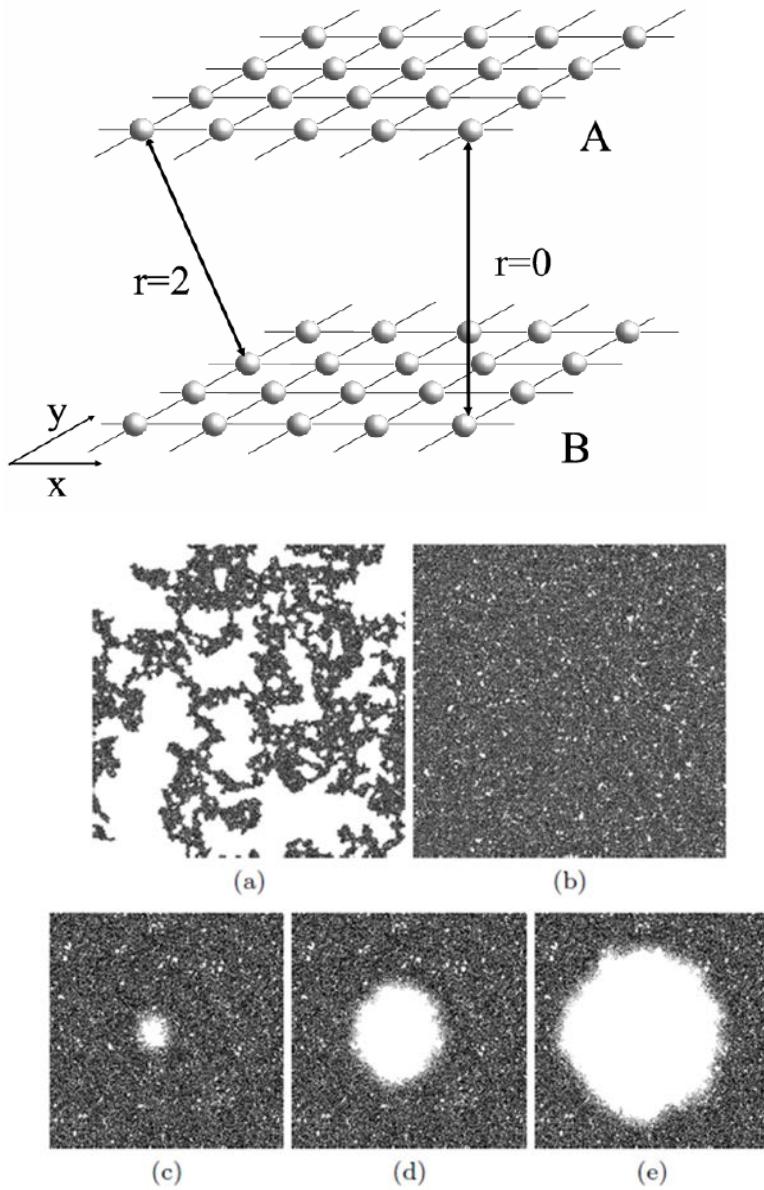


When connectivity links are limited  
in their length---same universality  
class as lattices!

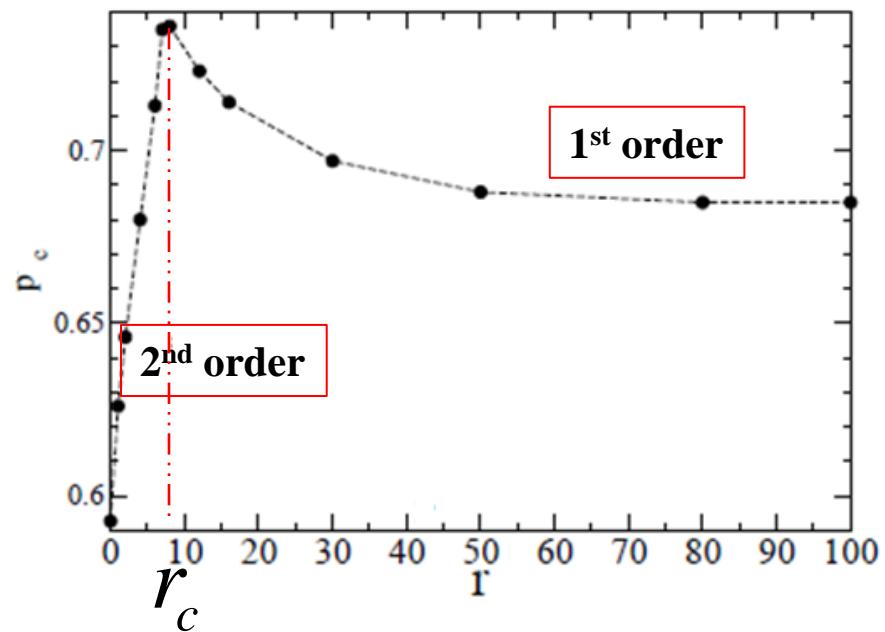
THREE DIFFERENT BEHAVIORS  
DEPENDING ON  $r$

Wei et al, PRL, 108, 228702 (2012)  
Bashan et al, Nature Physics (2013)

# Interdependent Spatially Embedded Networks



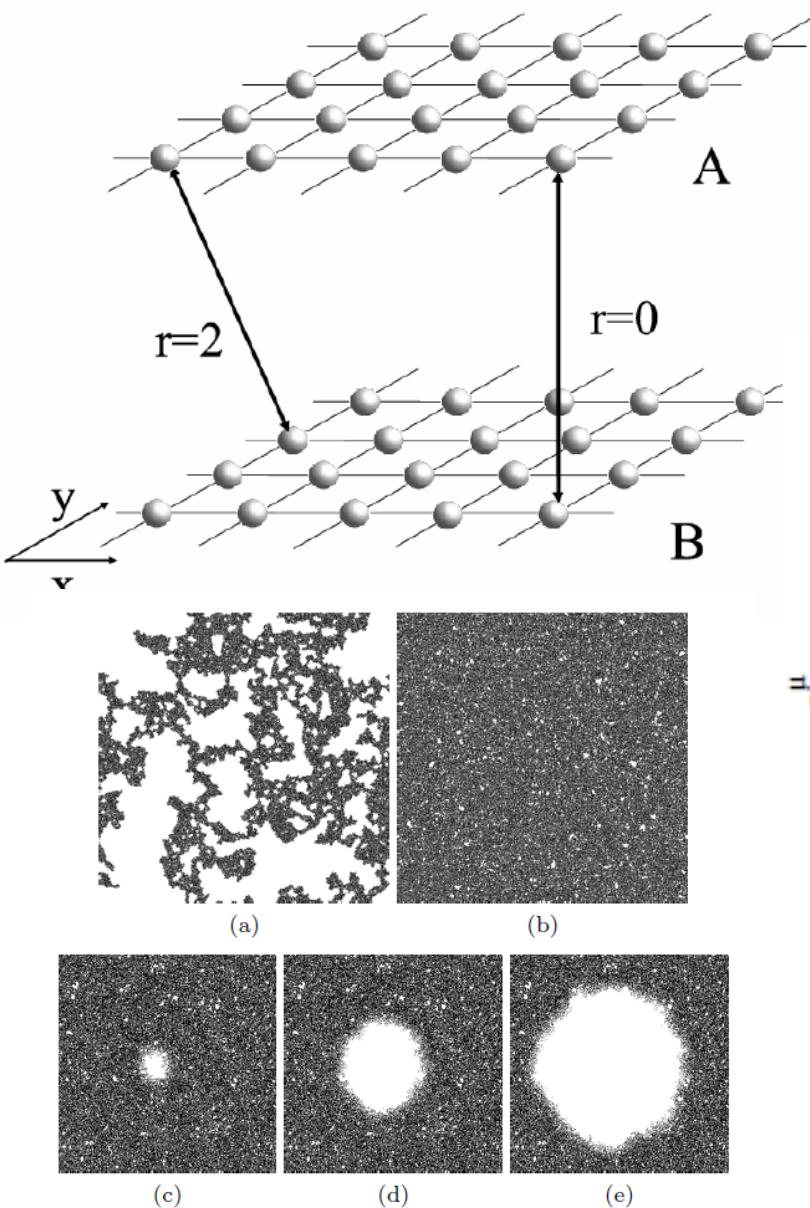
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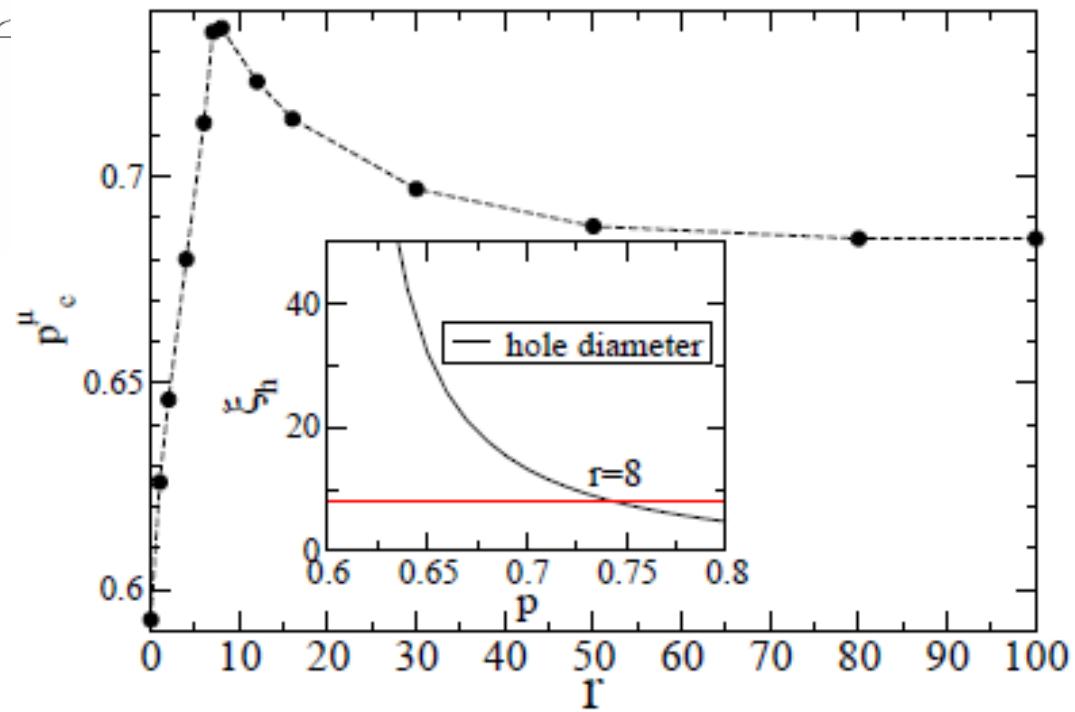
Wei et al, PRL, 108, 228702 (2012)

Bashan et al, Nature Phys. 9, 667 (2013)

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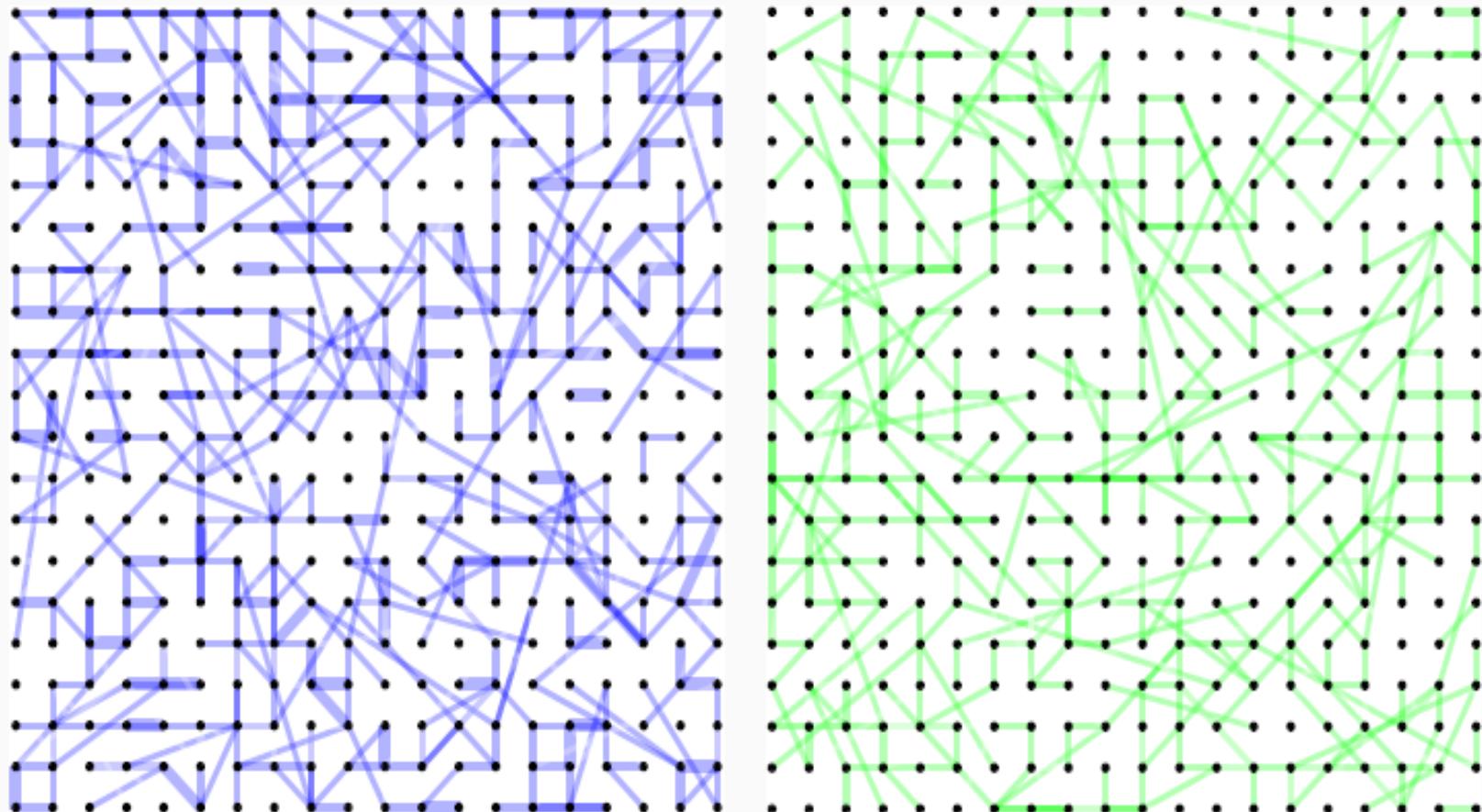


Many networks are spatially embedded:  
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Wei et al, PRL, 108, 228702 (2012)  
Bashan et al, Nature Physics (2013)

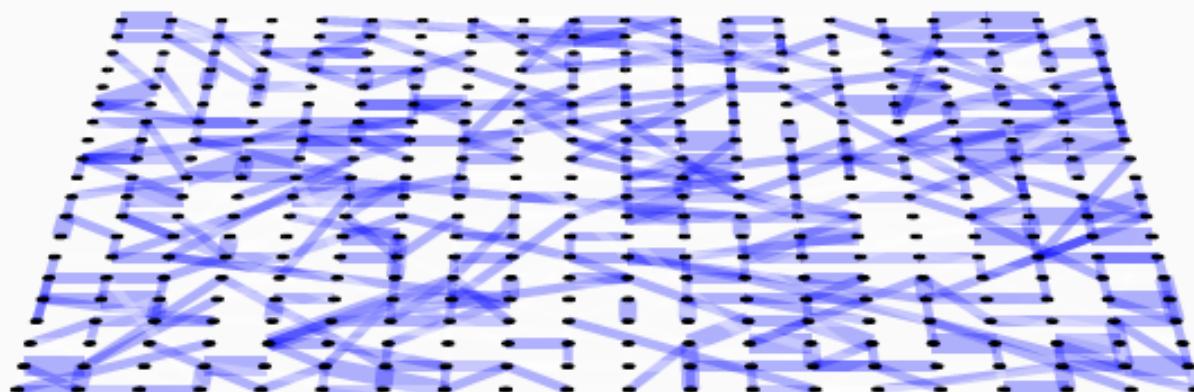
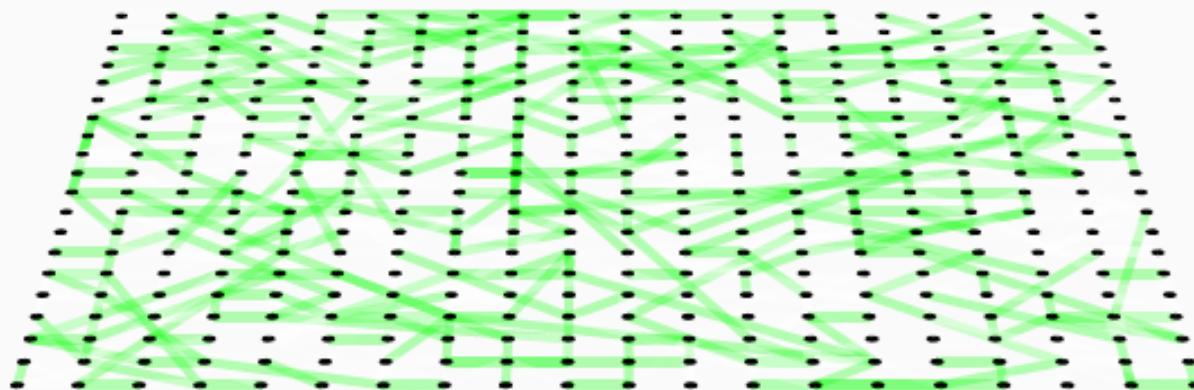
Each layer has links of characteristic length  $\zeta$ .



Danziger et al, arXiv:1505.01688 (2015)

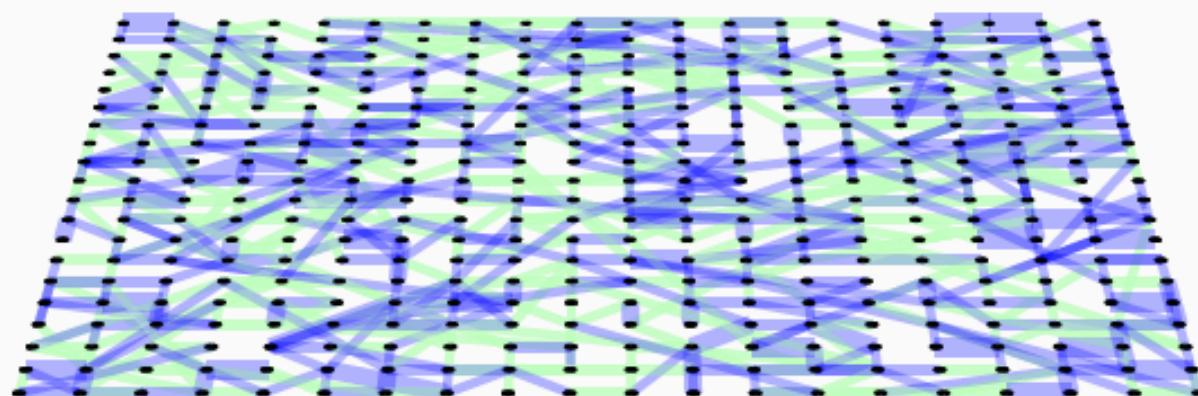
## SPATIALLY EMBEDDED MULTIPLEX NETWORK

Nodes in the same location in each layer are united.

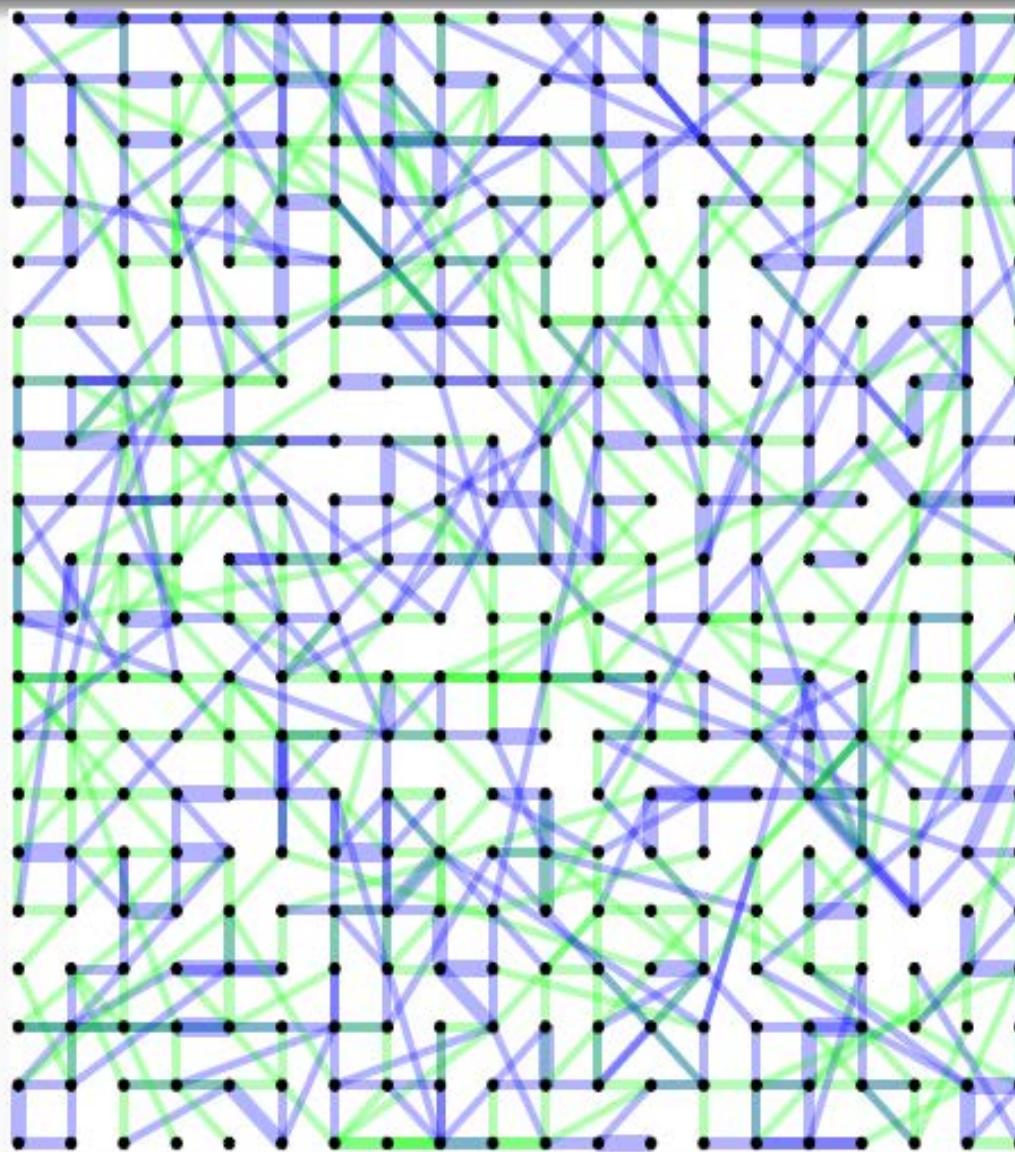


## SPATIALLY EMBEDDED MULTIPLEX NETWORK

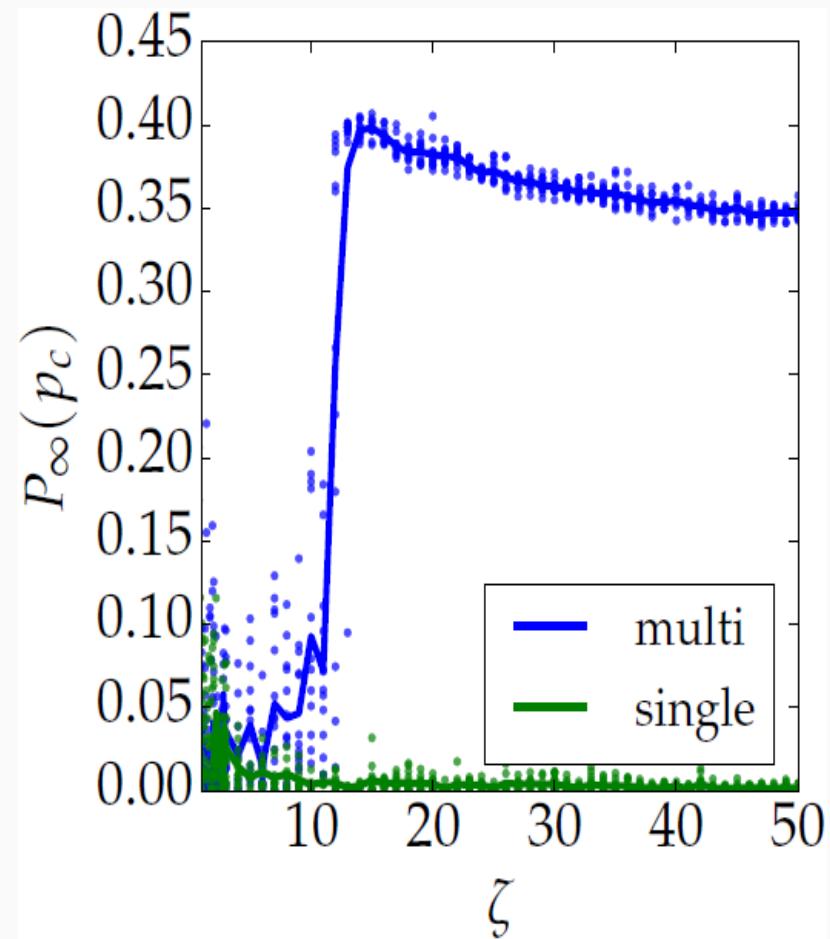
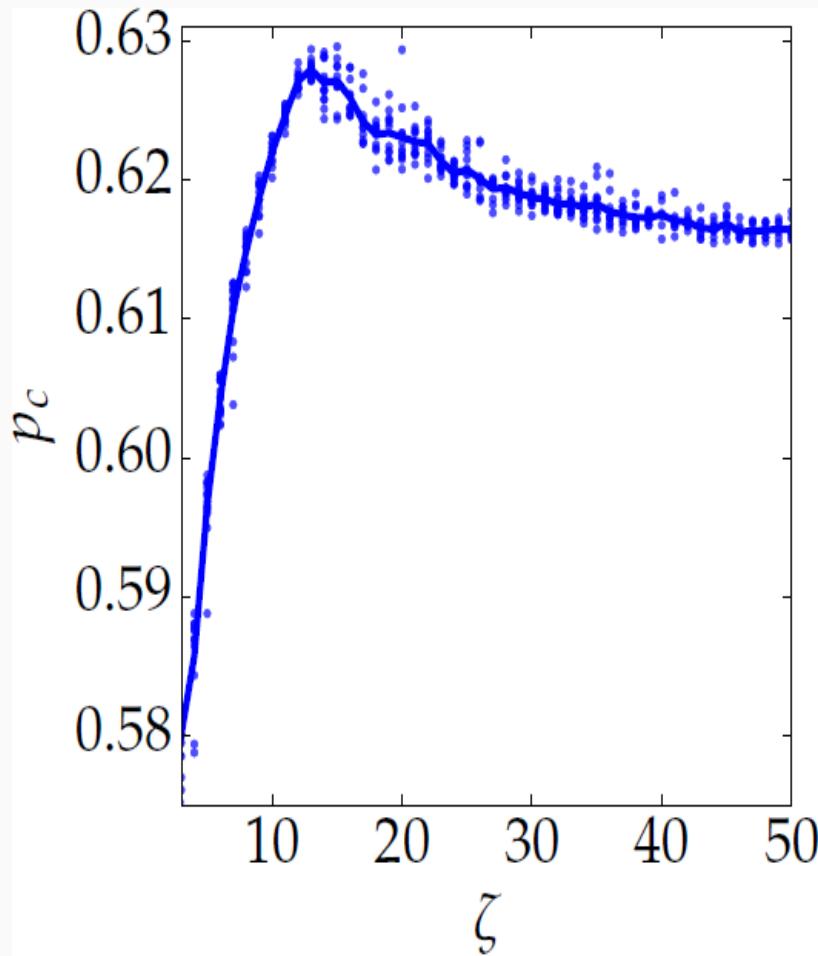
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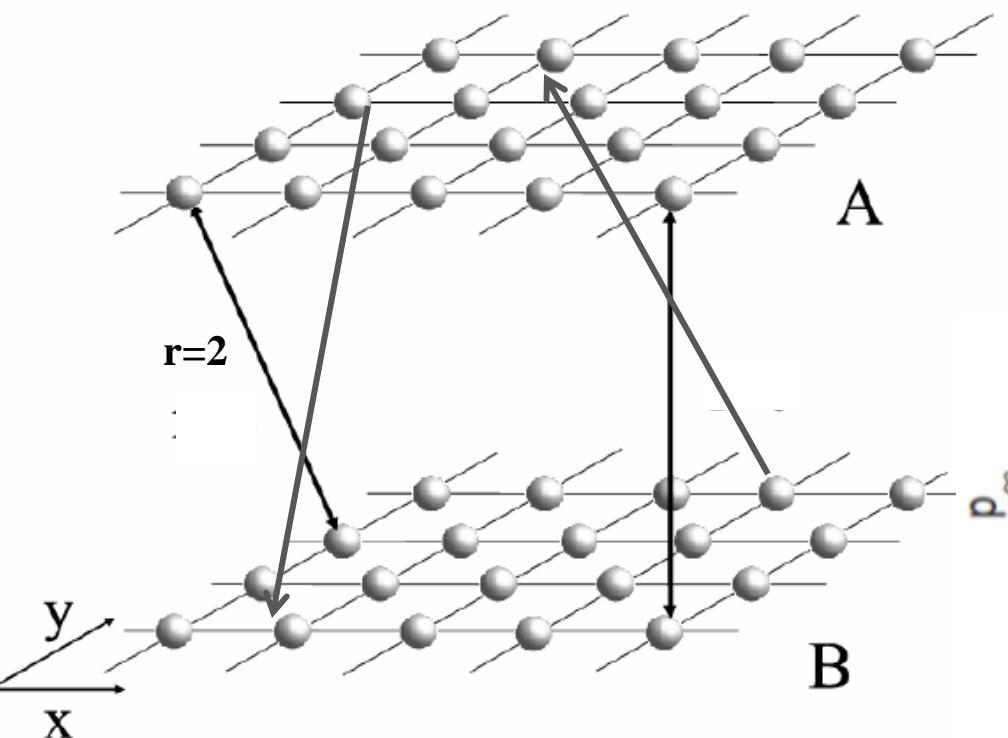


MULTIPLEX



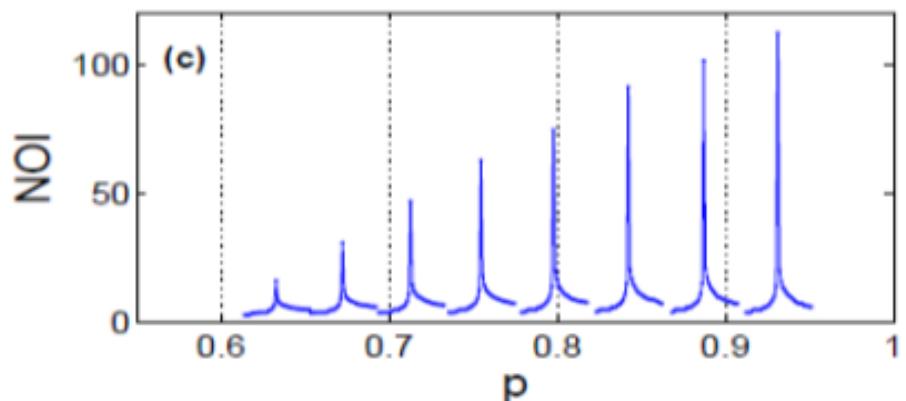
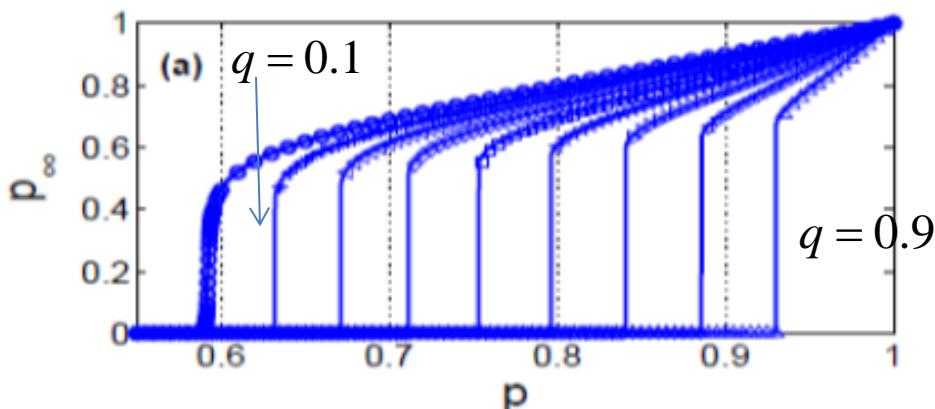
- Increased  $\zeta$  increases  $p_c \rightarrow$  spatiality *improves* robustness.
- At  $\zeta = \zeta_c \approx 11$ , the transition becomes abrupt.
- $p_c$  is maximal at  $\zeta_c$ : intermediate spatiality is the most vulnerable.

# Partially Interdependent Spatially Embedded Networks



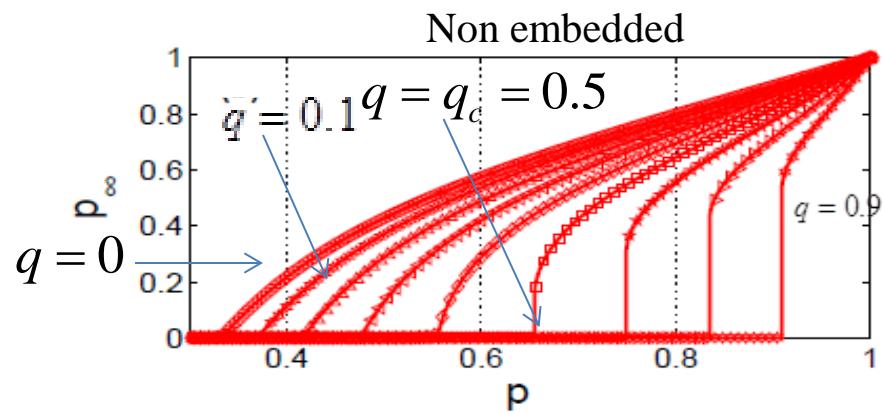
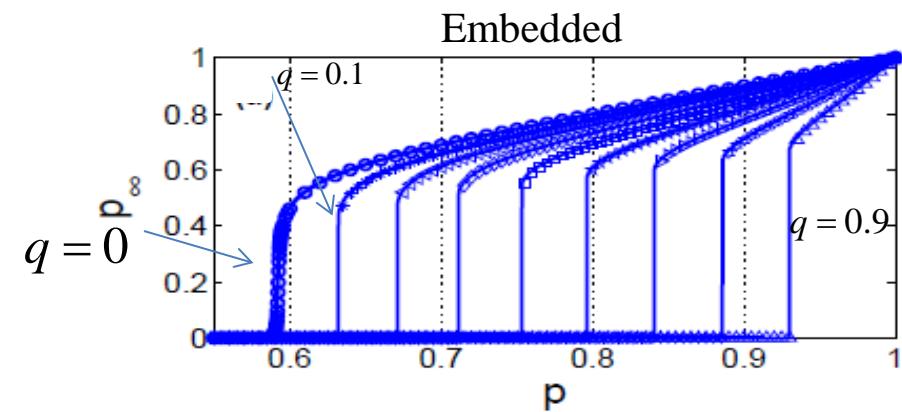
Many networks are spatially embedded:  
Internet, Power grid, Transportation etc

Bashan et al, Nature Physics ( 2013)



Theory (based on critical exponent):  
NO continuous transition  
for any  $q>0$ -extreme vulnerability!!

# Spatial embedded compared to random coupled networks when q changes



EXTREMELY VULNERABLE!!

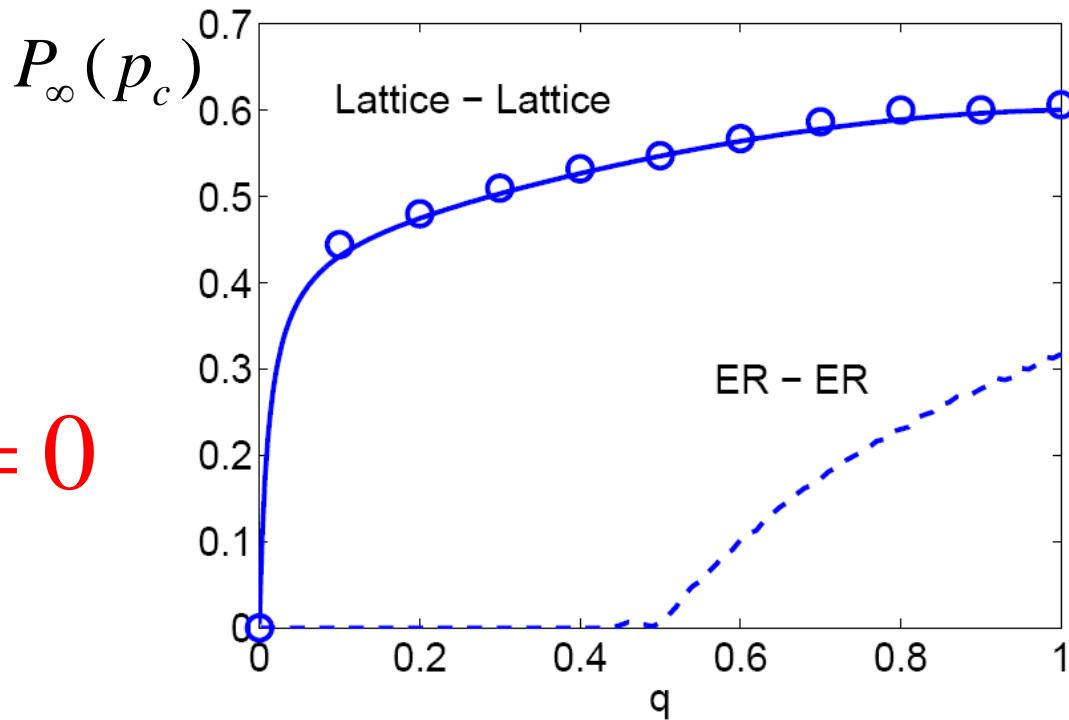
$$1 = p_c q_c P'_\infty(p_c)$$

$$P_\infty \sim (p - p_c)^\beta$$

$$\beta = 5/36 < 1 \text{ for } d=2$$

For ER and  $d=6$ ,  $\beta=1$

$$q_c = 0$$



Bashan et al

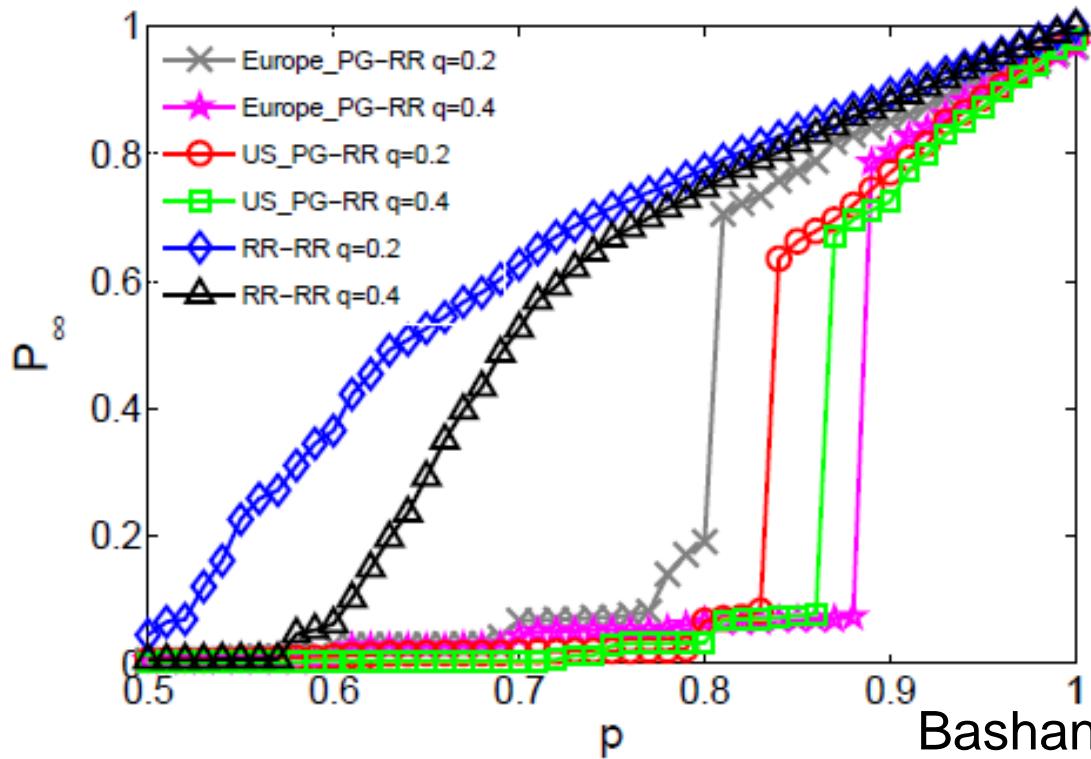
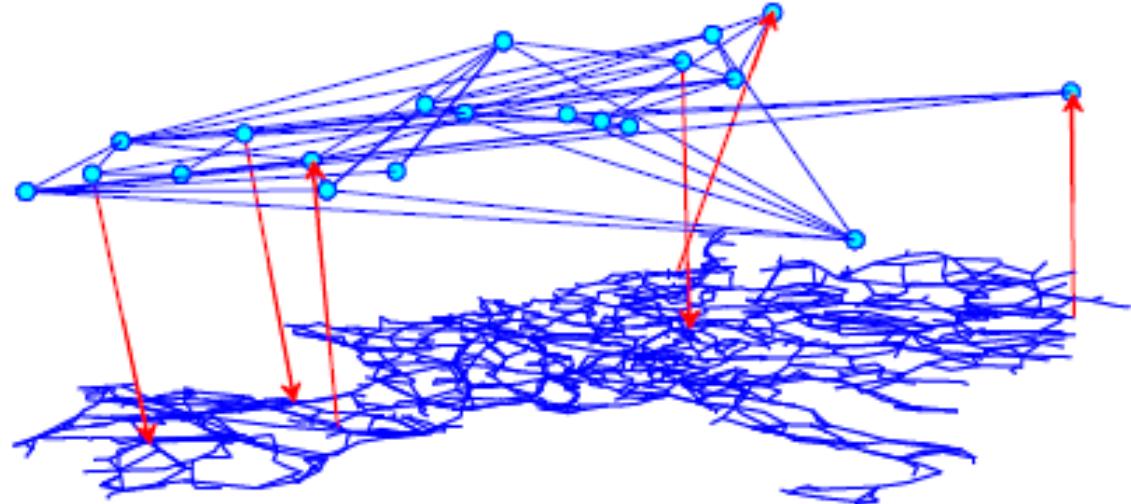
<http://arxiv.org/abs/1206.2062>

Nature Physics, (2013)

Message: our world is extremely unsafe!-no safe zone!

# Experimental test on real spatial embedded coupled networks

Interdependent European  
Communication Network and  
Power Grid

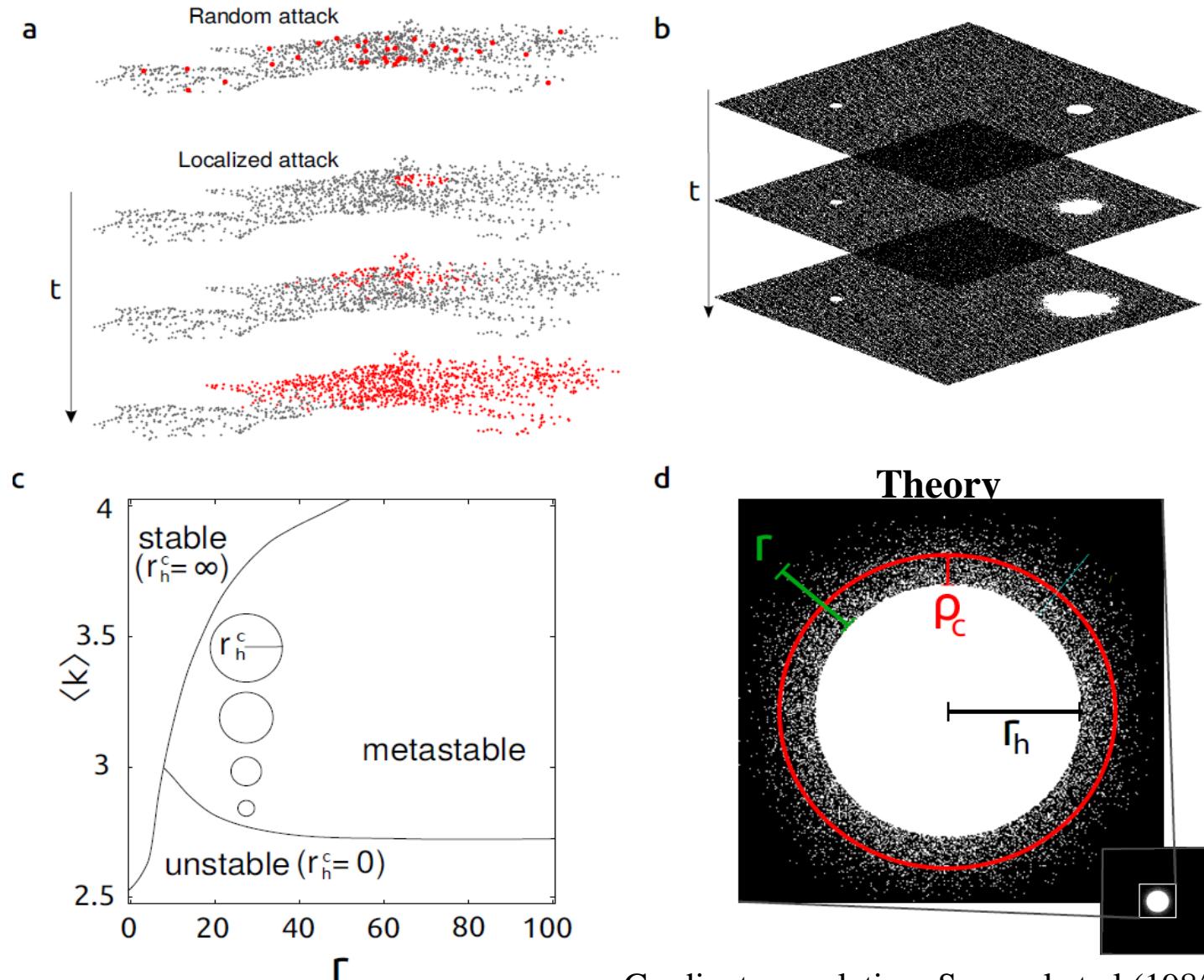


Results for European and US  
Interdependent  
Communication Networks and  
Power Grids

# New percolation-localized attacks

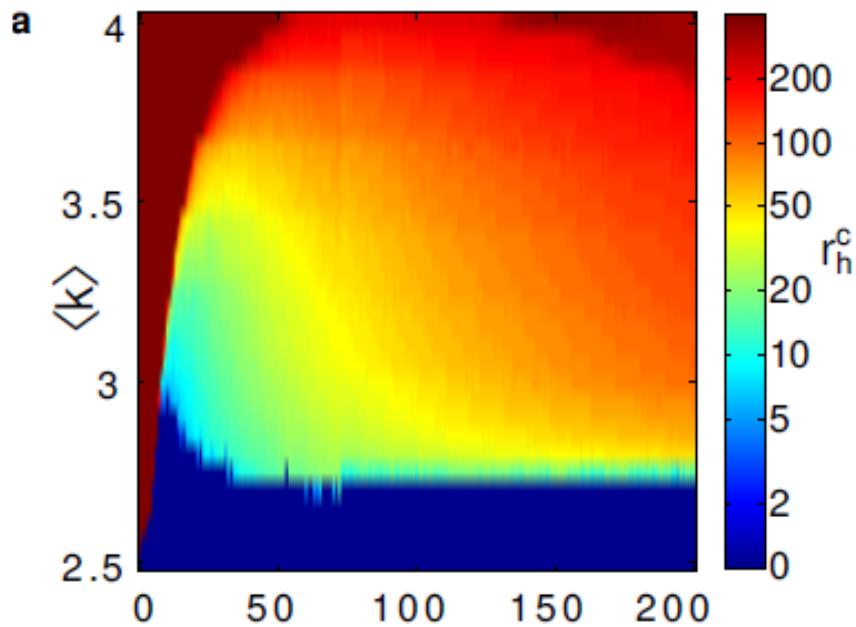
Localized attacks on spatially embedded systems with dependencies: critical size attack

Y. Berezin et al., .  
arXiv:1310.0996

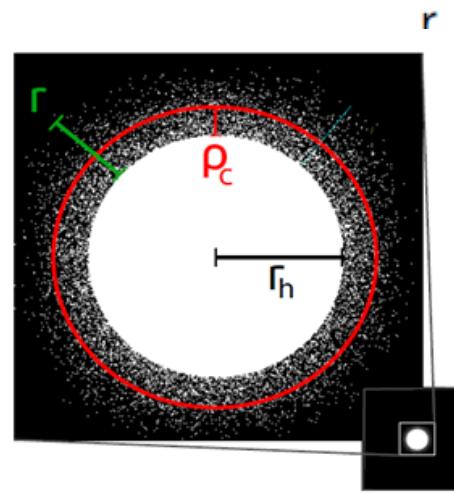
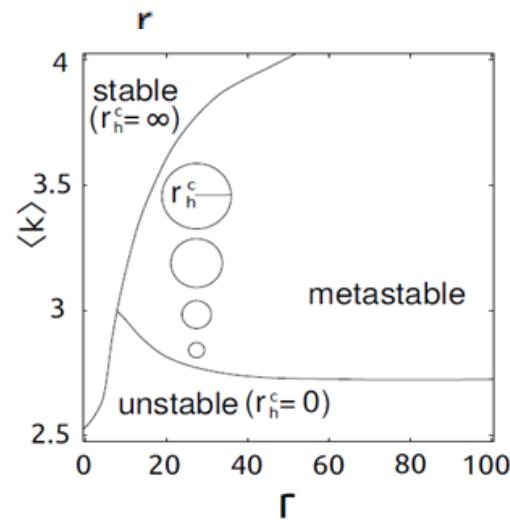
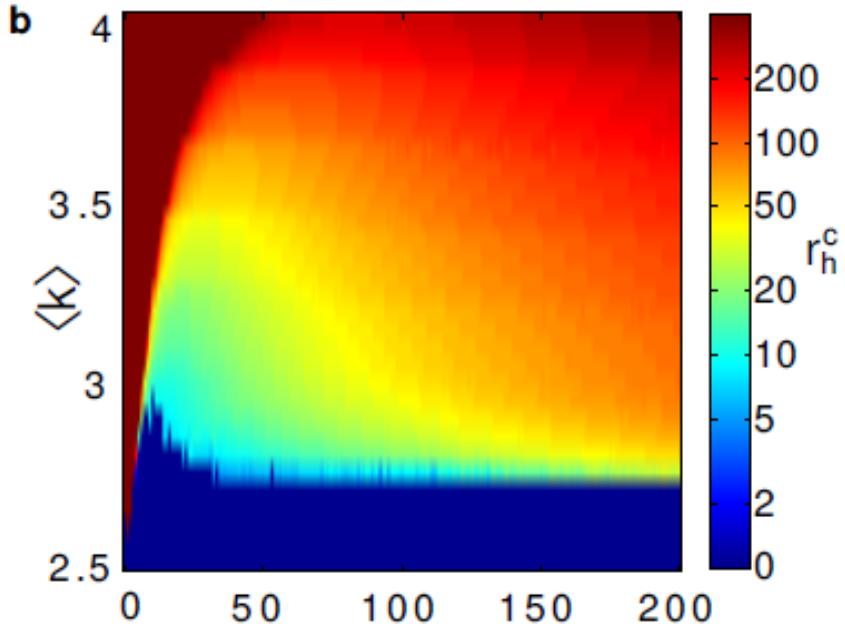


# New percolation-localized attacks

Simulation Results



Analytical Results



# Summary and Conclusions

- First statistical physics approach for robustness of Networks of Interdependent Networks—cascading failures
- New paradigm: abrupt collapse compared to continuous in single network
- Generalization to “Network of Networks”: n interdependent networks- 50y of graph theory and percolation is only a limited case!
- Spontaneous recovery of systems of systems
- Spatial embedding extremely risky  $q_c = 0$

Rich problem: different types of networks and interconnections.

Buldyrev et al., NATURE (2010)

Parshani et al., PRL (2010)

Gao et al, PRL (2011)

Parshani et al, PNAS (2011)

Wei et al, PRL (2012)

Gao et al., Nature Phys. (2012)

Bashan et al, Nature Phys. (2013)

Danziger et al, ArXiv: 1505.01688 (2015)

Majdamzic et al Nat. Phys. (2014); arXiv:1502.00244V2 (2015)

