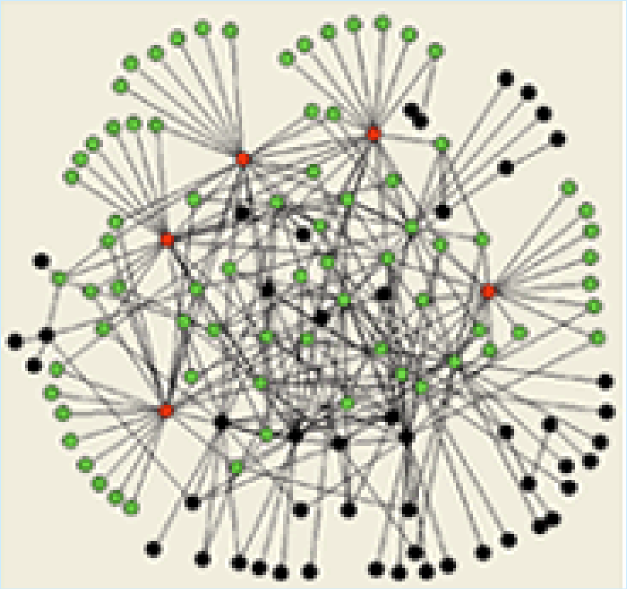
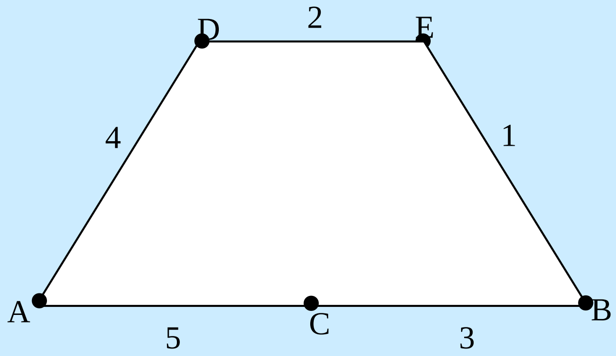
Percolation in Complex Networks: Optimal Paths and Minimum Spanning Trees



Optimal Distance - Disorder



Path from
A to **B**

l_{min}= 2(ACB)

l_{opt}= 3(ADEB)

$\boldsymbol{w_i}$ = weight = price, quality, time…..

$$\sum_i \boldsymbol{w_i} = \text{ minimal } \Rightarrow \text{ optimal path}$$

Weak disorder (WD) – all $\boldsymbol{w_i}$ contribute to the sum (narrow distribution)

Strong disorder (SD)– a single term dominates the sum (broad distribution)

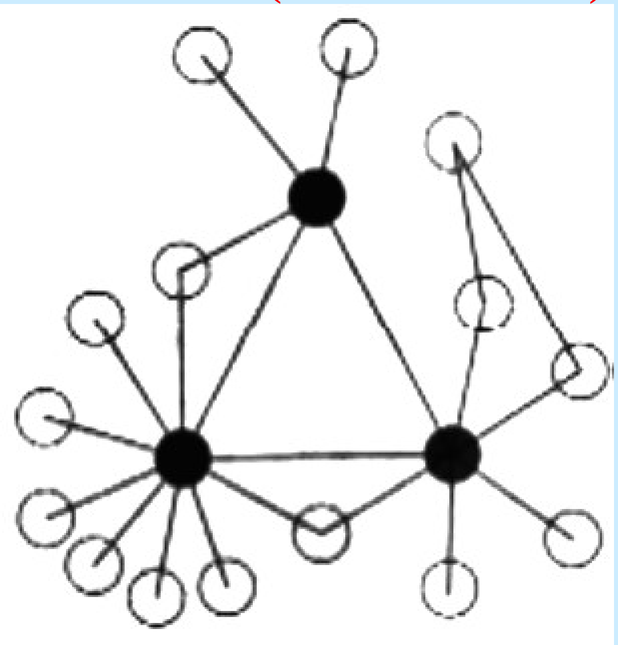
SD – example: Broadcasting video over the Internet,

a transmission at constant high rate is needed.

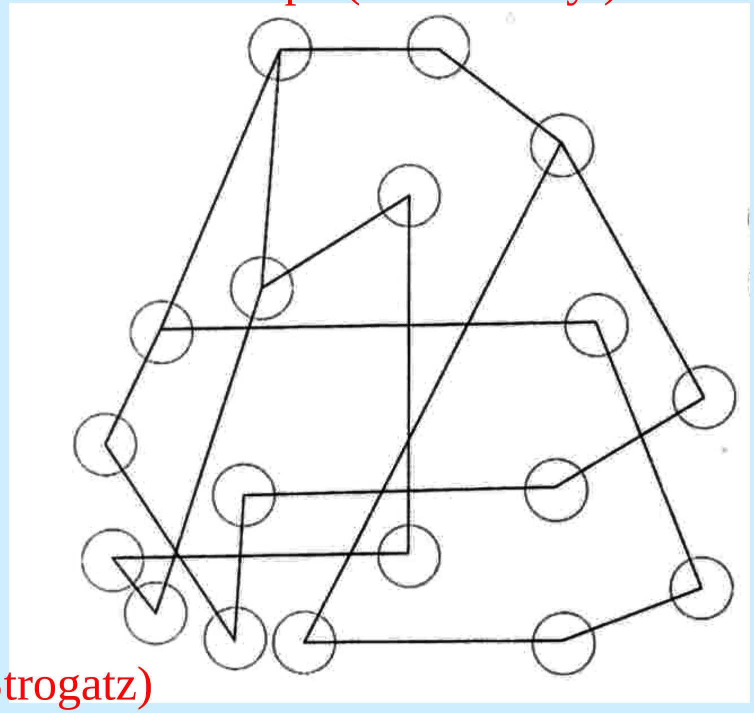
The narrowest band width link in the path

between transmitter and receiver controls the rate.

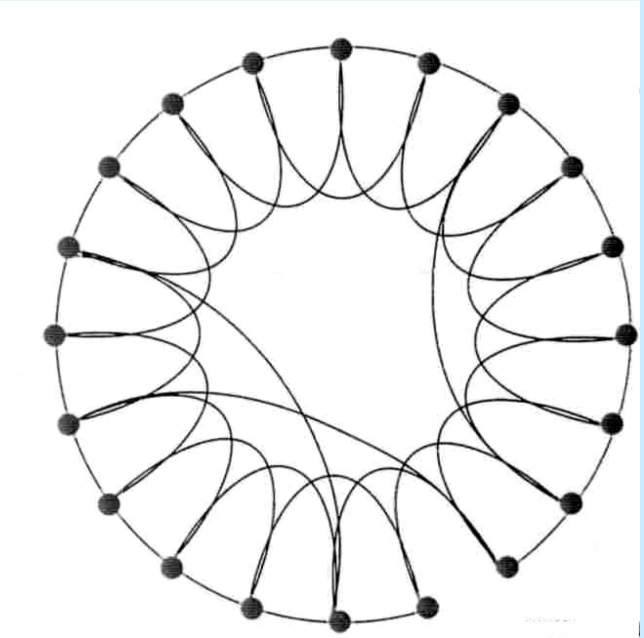
Scale Free (Barabasi-Albert)



Random Graph (Erdos-Renyi)



Small World (Watts-Strogatz)



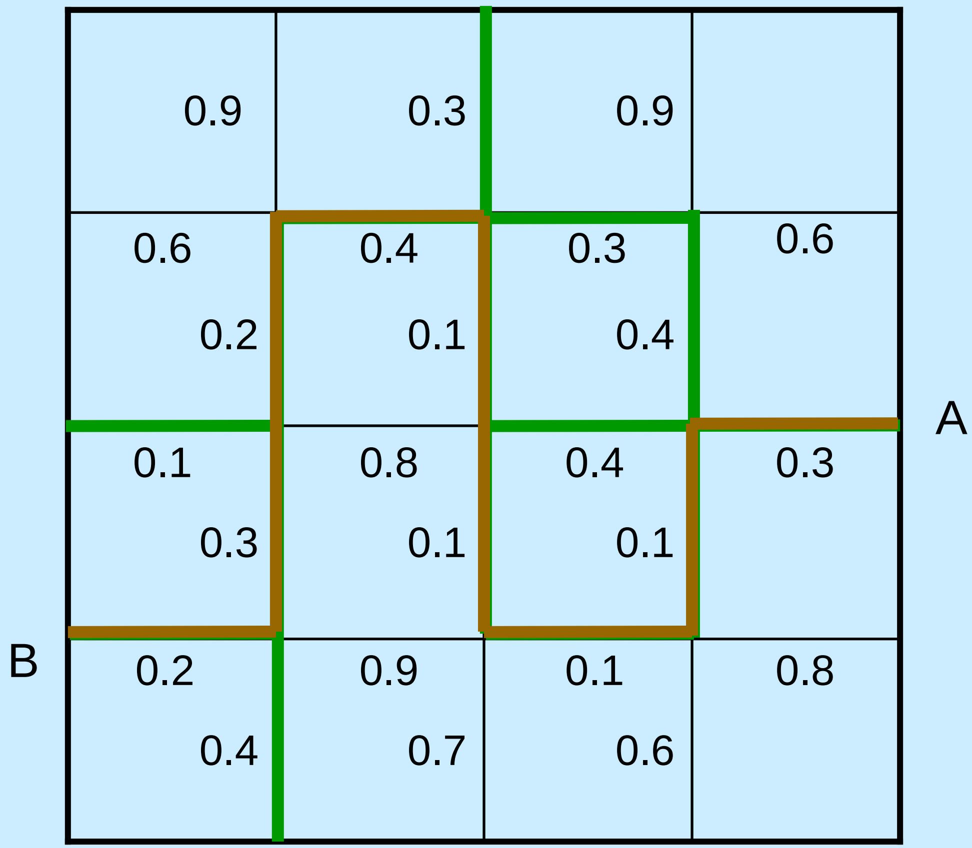
$Z = 4$

Optimal Path

$$\ell_{opt} \sim r^{dopt} \qquad \ell \sim r^{d\,\min}$$

d	$d_{\min}$	SD d_{opt}	WD d_{opt}
2	1.13	1.22	1
3	1.37	1.41	1
4	1.59		1
5			1
6	2	2	1

Strong Disorder and Percolation



Optimal path – weak disorder

Random Graphs and Watts Strogatz Networks

$$\boldsymbol{l}_{\min} \sim \log \boldsymbol{N}$$

$$l_{opt} \sim \log N$$

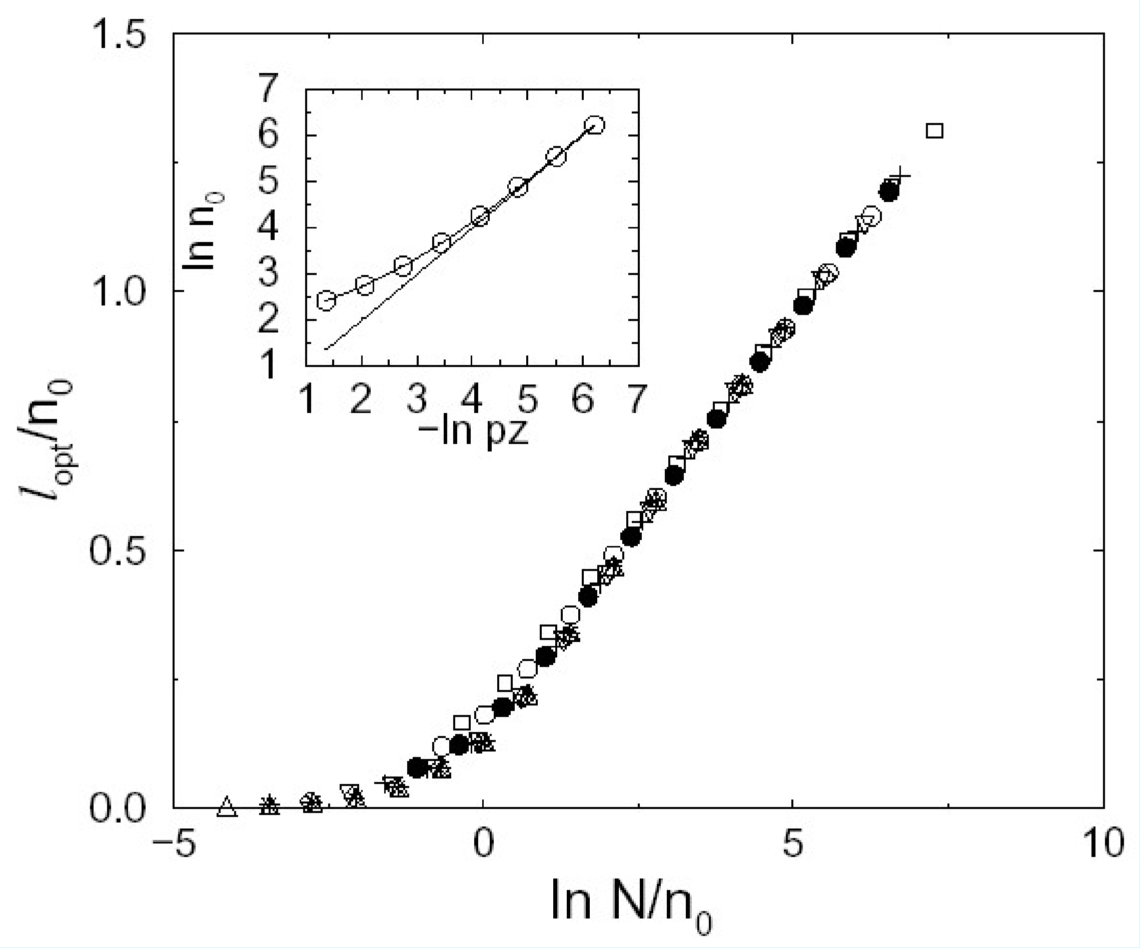
$$n_0 \propto \frac{1}{pz}$$

Typical short range neighborhood

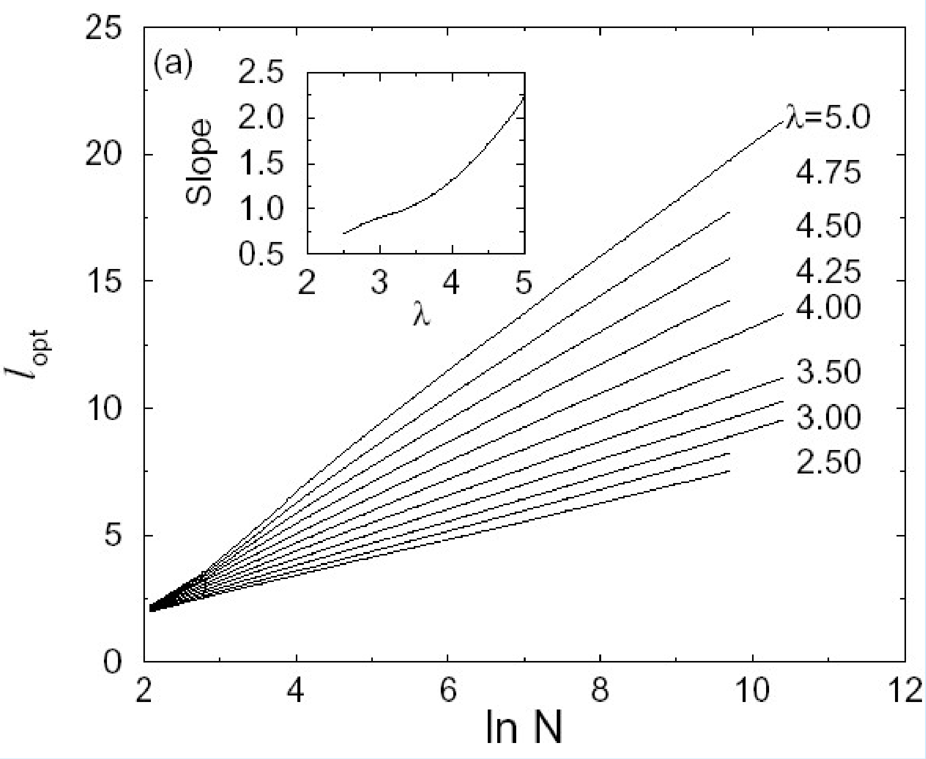
Crossover from large to small world

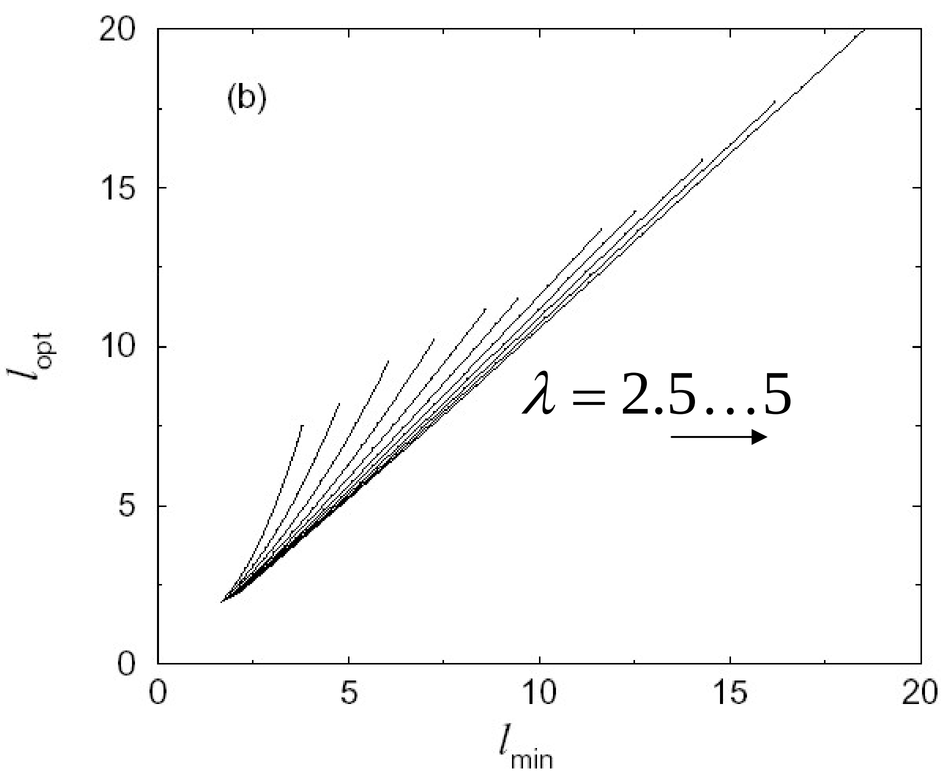
For $N / n_0 < 1$

$$l_{opt} \sim N \sim e^{\log N / n_0}$$



Scale Free – Optimal Path – Weak disorder





$$For \quad \lambda > 3$$

$$l_{opt} \sim A(\lambda)\log N$$

$$l_{\min} \sim \log N$$

$$Thus \quad \boxed{l_{opt} \sim l_{\min}}$$

$$For \quad 2 < \lambda < 3$$

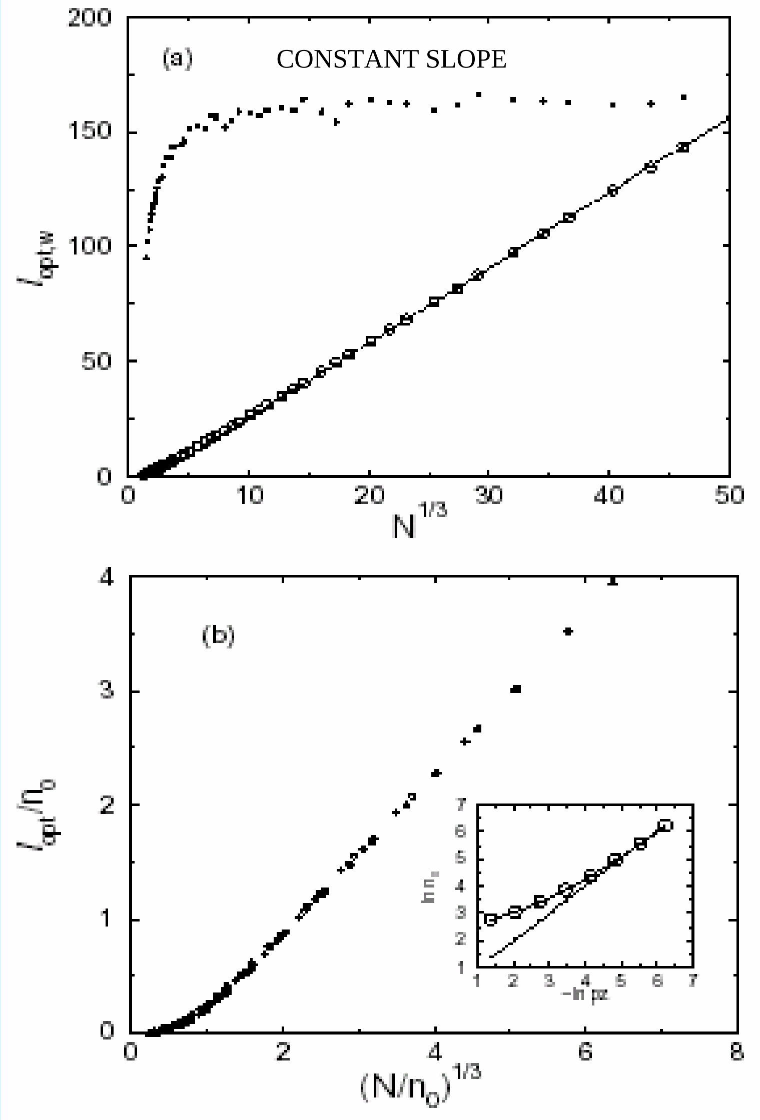
$$l_{opt} \sim \log N$$

$$l_{\min} \sim \log\log N$$

$$Thus \quad \boxed{l_{opt} \sim \exp(l_{\min})}$$

Optimal path – strong disorder

Random Graphs and Watts Strogatz Networks



N – total number of nodes

$l_{opt} \sim N^{1/3}$ Analytically and Numerically

LARGE WORLD!!

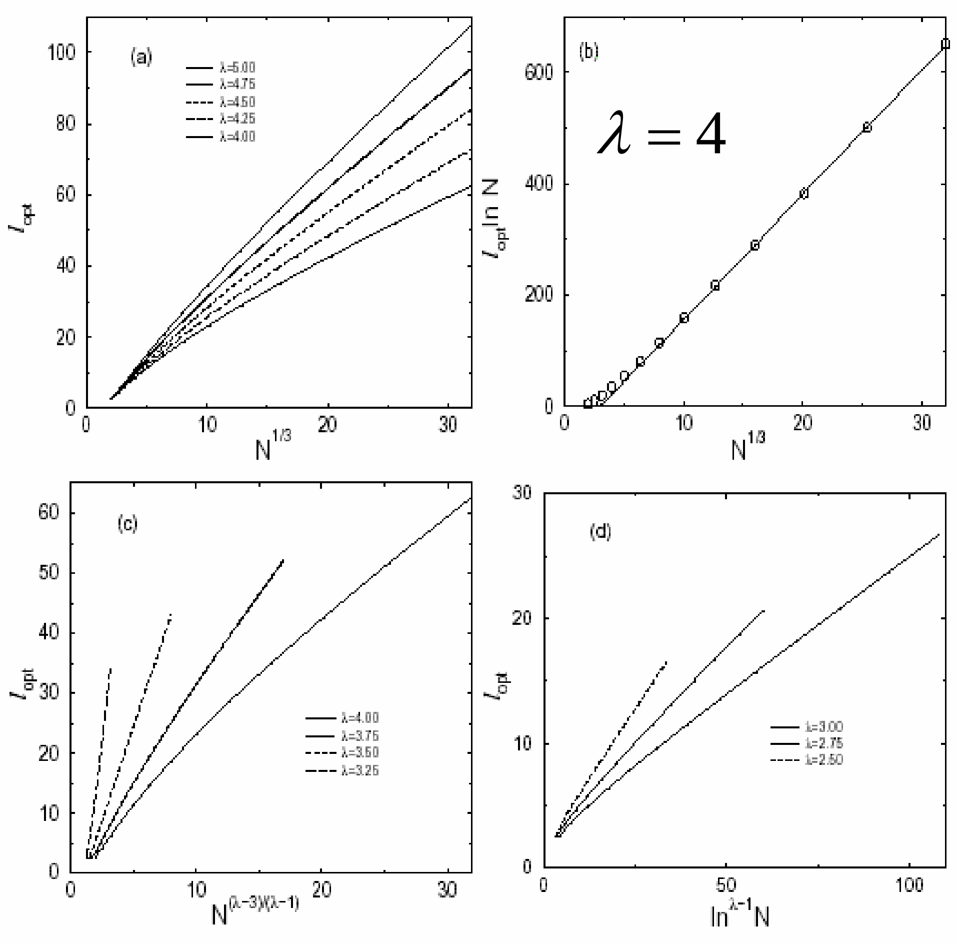
Compared to the diameter or average shortest path or weak disorder

$l_{\min} \sim \log N$ (small world)

n_0 - typical range of neighborhood without long range links

$\frac{N}{n_0}$ - typical number of nodes with long range links

Scale Free – Optimal Path



Strong Disorder

Theoretically + Numerically

$$l_{opt} \sim \begin{cases} N^{(\lambda-3)/(\lambda-1)} & 3<\lambda<4 \\ N^{1/3}\log N & \lambda = 4 \\ N^{1/3} & \lambda>4 \end{cases}$$

LARGE WORLD!!

Numerically $\quad l_{opt} \sim \log^{\lambda-1} N \quad 2<\lambda<3$

SMALL WORLD!!

Weak Disorder

$$l_{opt} \sim \log N \quad \textit{for all } \lambda$$

Diameter – shortest path

$$l_{\min} \sim \begin{cases} \log N & \lambda>3 \\ \log N / \log\log N & \lambda = 3 \\ \log\log N & 2<\lambda<3 \end{cases}$$

Braunstein, Buldyrev, Cohen, Havlin, Stanley, Phys. Rev. Lett. 91, 247901 (2003); Cond-mat/0305051

Theoretical Approach – Strong Disorder

(i) Distribute random numbers 0<u<1 on the links of the network.

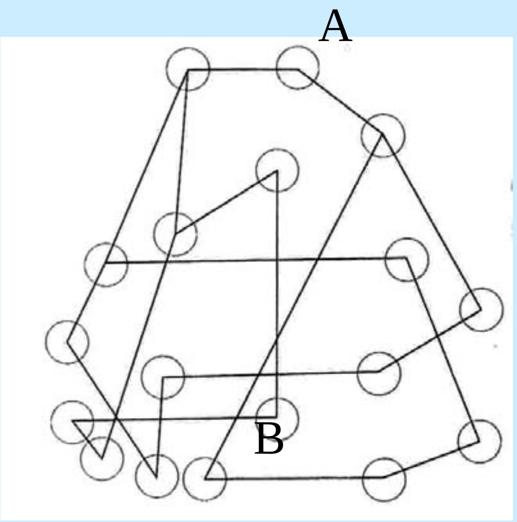
(ii) Strong disorder represented by $\varepsilon_i = \exp(au_i)$ with $a \to \infty$.

(iii) The largest u_i in each path between two nodes dominates the sum.

(iv) The optimal path is the path with the min-max

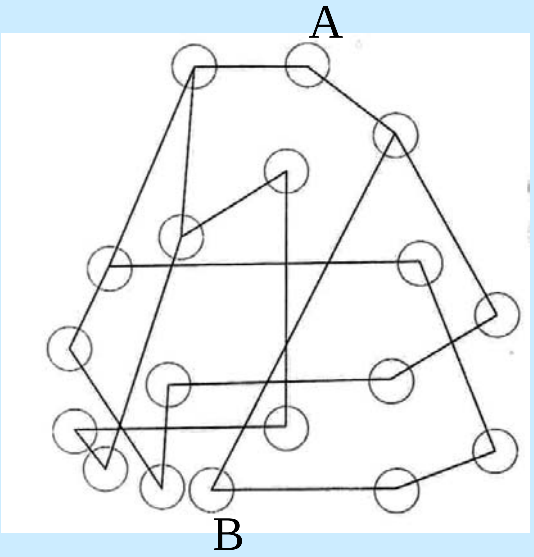
(v) Percolation exists if we remove all links with $u_i > 1 - p_c$

(vi) The optimal path must therefore be on the percolation cluster at criticality.



What do we know about percolation clusters at criticality in networks?

Theoretical Approach – Strong Disorder Conclusions



(i) Percolation on random networks is like percolation in $d \to \infty$ or $d = d_c$.

(ii) Since loops can be neglected the optimal path can be identified with the shortest path on percolation-only a single path exist between any nodes.

Calculate the length of shortest path:

Mass of infinite cluster $S \sim R^{d_f}$ where $N \sim R^d$

For ER $d_c = 6$, $S \sim N^{d_f/d} = N^{4/6} = N^{2/3}$ (see also Erdos-Renyi, 1960)

From percolation $S \sim l^2, (d_l = 2)$

Thus, for ER, WS and SF with $\lambda > 4$: $l_{opt} \sim l \sim S^{1/2} \sim N^{1/3}$

For SF with $3 < \lambda < 4$ $d_c, d_f,$ and d_l change due to novel topology: $l_{opt} \sim N^{(\lambda-3)/(\lambda-1)}$

Fractal Dimensions

From the behavior of the critical exponents the fractal dimension of scale-free graphs can be deduced.

Far from the critical point - the dimension is infinite - the mass grows exponentially with the distance.

At criticality - the dimension is finite for $\lambda>3$.

Short path dimension:

$$S \sim \ell^{d_\ell}$$

$$d_l = \begin{cases} \frac{\lambda-2}{\lambda-3} & \lambda < 4 \\ 2 & \lambda \geq 4 \end{cases}$$

Fractal dimension:

$$S \sim R^{d_f}$$

$$d_f = \begin{cases} 2\frac{\lambda-2}{\lambda-3} & \lambda < 4 \\ 4 & \lambda \geq 4 \end{cases}$$

Embedding dimension:

(upper critical dimension)

$$d_c = \begin{cases} 2\frac{\lambda-1}{\lambda-3} & \lambda < 4 \\ 6 & \lambda \geq 4 \end{cases}$$

Random Graphs – Erdos Renyi(1960)

Largest cluster at criticality

$$S \sim N^{\frac{2}{3}}$$

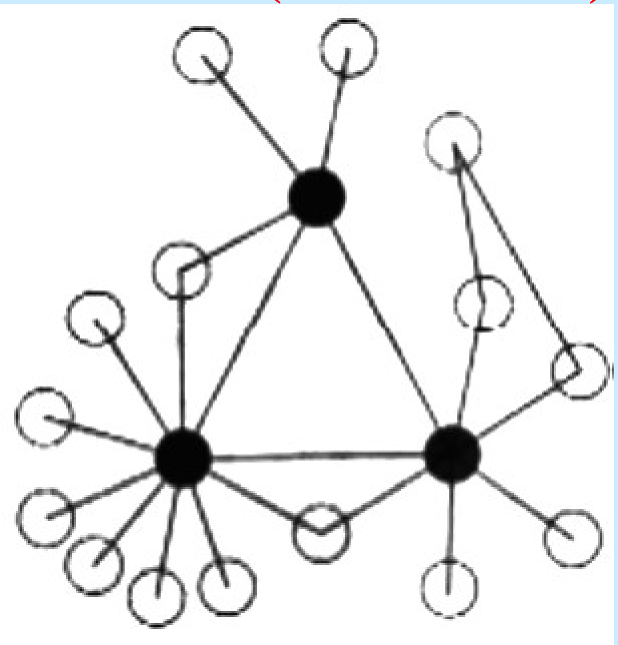
Scale Free networks

$$S \sim R^{d_f} \sim N^{\frac{d_f}{d_c}} \sim N^{\frac{\lambda-2}{\lambda-1}} \qquad \lambda \leq 4$$

$$S \sim N^{\frac{2}{3}} \qquad \lambda \geq 4$$

The dimensionality of the graphs depends on the distribution!

Scale Free (Barabasi-Albert)



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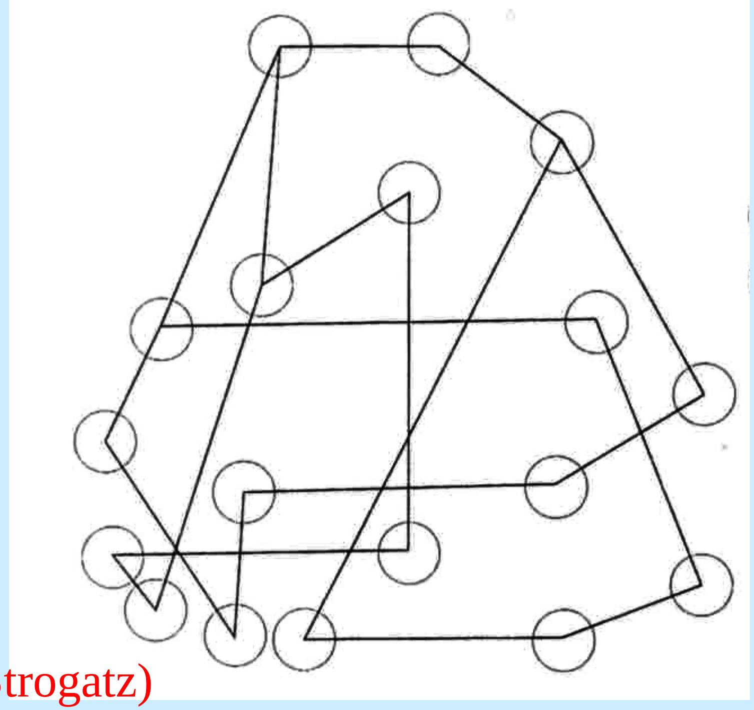
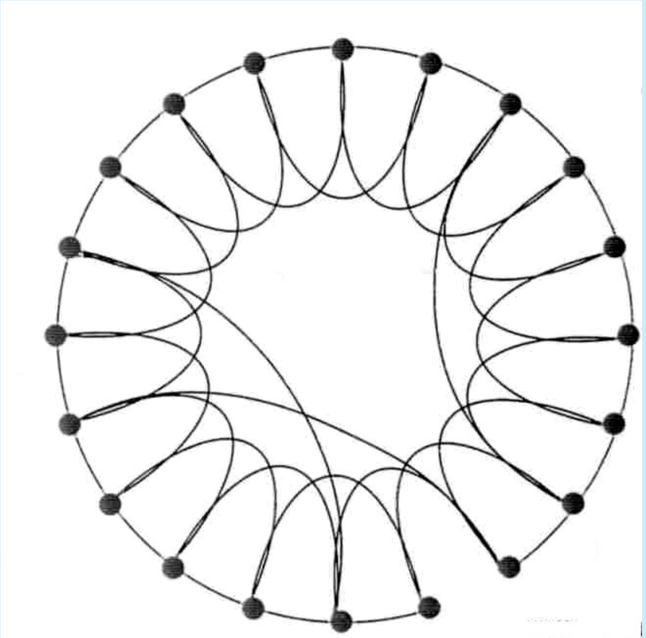


$$P(k) = Ak^{-\lambda}$$

Small World (Watts-Strogatz)



$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

Z = 4

Shortest Paths in Scale Free Networks

$$P(k) \sim k^{-\lambda}$$

$$d = const. \qquad \lambda = 2$$

Ultra Small World

$$\boxed{d = \log\log N \qquad 2 < \lambda < 3}$$

$$d = \frac{\log N}{\log\log N} \qquad \lambda = 3$$

(Bollobas, Riordan, 2002)

Small World

$$d = \log N \qquad \lambda > 3$$

(Bollobas, 1985)
(Newman, 2001)

Same as for ER and WS

Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks

eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) Chap.4

Cohen, Havlin Phys. Rev. Lett. 90, 58701(2003)

Also by: Dorogovtsev, Mendes et al (2002), Chung and Lu (2002)

Optimal path – weak disorder

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$$l_{opt} \sim \log N$$

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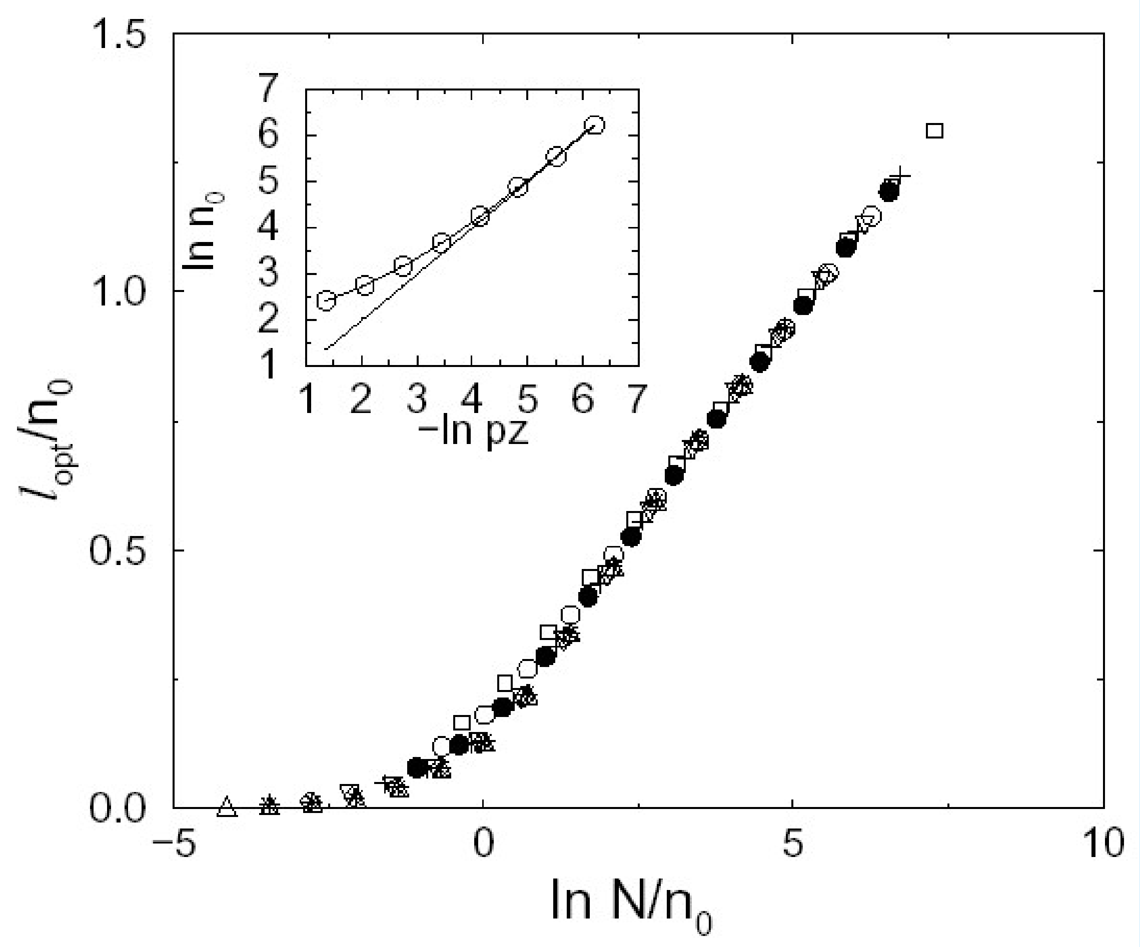

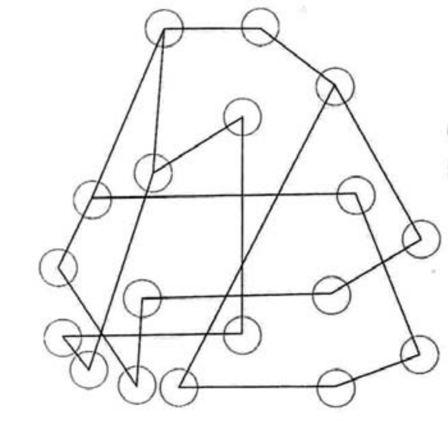

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Since $\ell \sim r^2$ it follows that $S \sim \ell^2, (d_\ell = 2)$

Thus, for ER, WS and SF with $\lambda > 4$: $l_{opt} \sim l \sim S^{1/2} \sim N^{1/3}$

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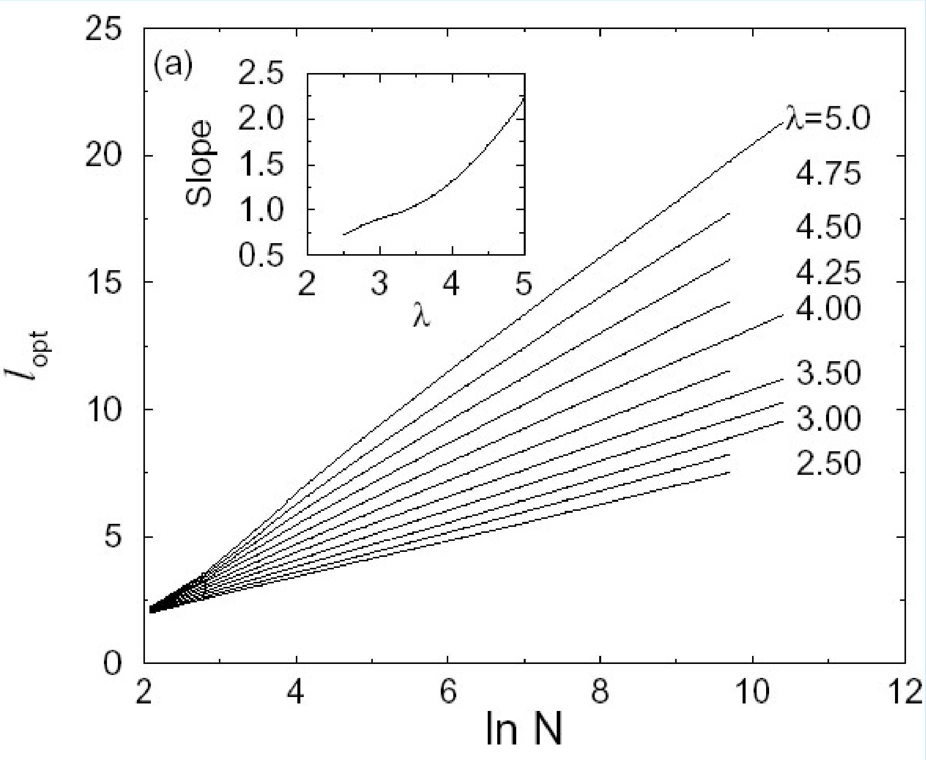
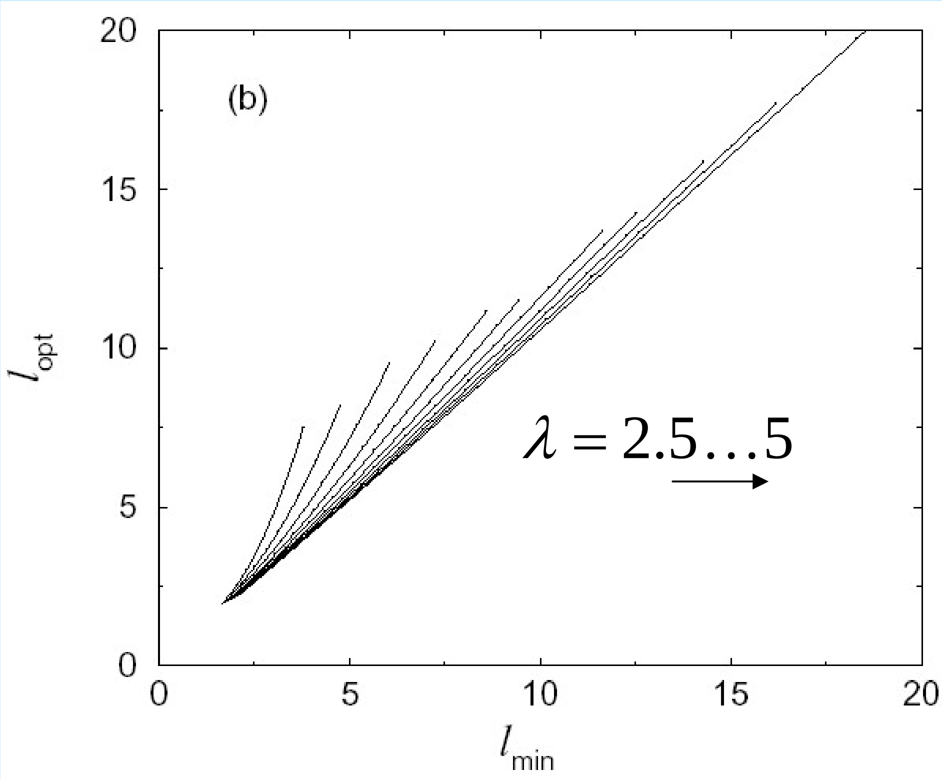
Scale Free – Optimal Path – Weak disorder





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$$l_{\min} \sim \log N$$

$$Thus \; \boxed{l_{opt} \sim l_{\min}}$$

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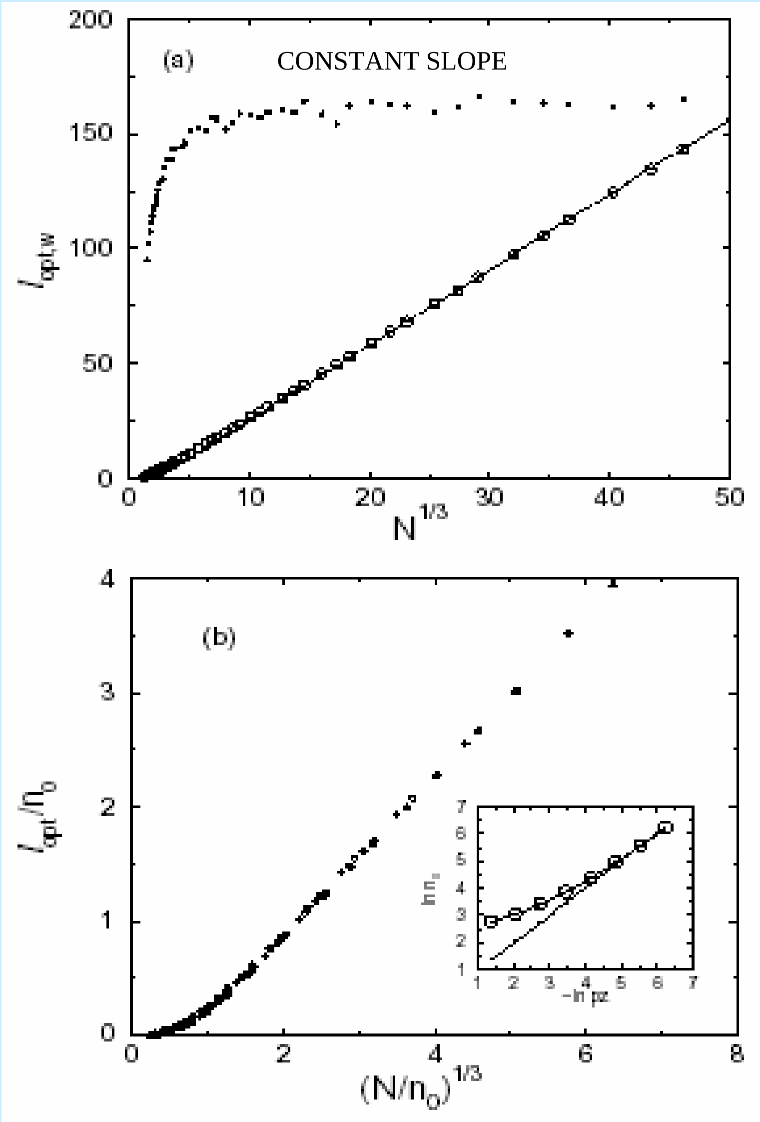
$$l_{\min} \sim \log \log N$$

$$Thus \; \boxed{l_{opt} \sim \exp(l_{\min})}$$

Optimal path – strong disorder

Random Graphs and Watts Strogatz Networks



N – total number of nodes

$\boldsymbol{l_{opt}} \sim \boldsymbol{N}^{1/3}$ Analytically and Numerically

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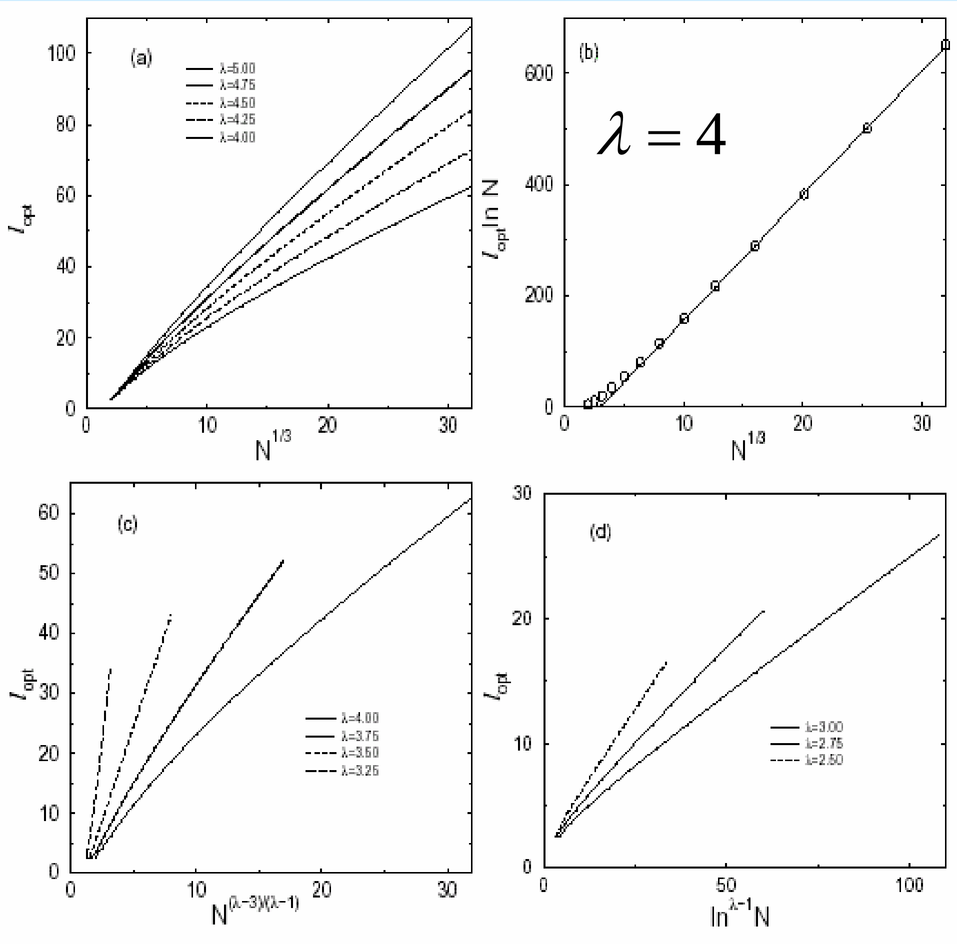
$\dfrac{\boldsymbol{N}}{\boldsymbol{n}_0}$ - typical number of nodes with long range links

Scale Free – Optimal Path



Strong Disorder

Theoretically + Numerically

$$l_{opt} \sim \begin{cases} N^{(\lambda-3)/(\lambda-1)} & 3 < \lambda < 4 \\ N^{1/3} \log N & \lambda = 4 \\ N^{1/3} & \lambda > 4 \end{cases}$$

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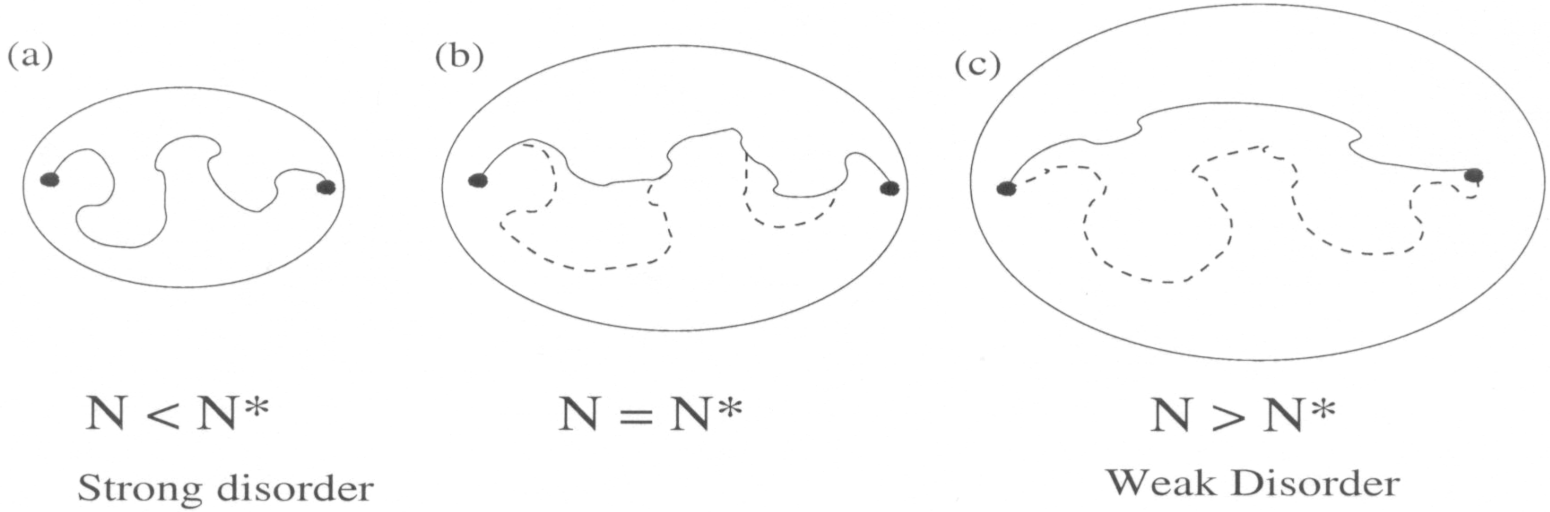
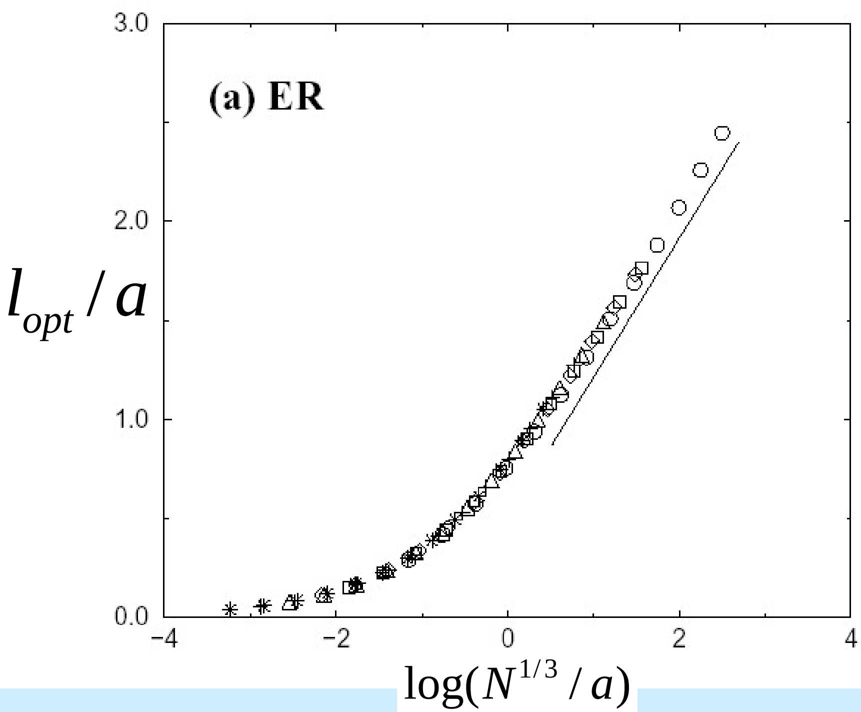
Braunstein et al,
Phys. Rev. Lett. 91, 247901 (2003);
Cond-mat/0305051

Transition from weak to strong disorder

For a given disorder strength a $\quad (\varepsilon_i = \exp(a u_i))$





$$N^* = N^*(a)$$

$$l_{opt} \sim \log N \quad \text{for} \quad N^{1/3} > a$$

$$l_{opt} \sim \exp(\log N^{1/3}) \text{ for } N^{1/3} < a$$

Sreenivasan et al Phys. Rev. E Submitted (2004)

For details see POSTER

Conclusions and Applications

- Distance in scale free networks λ<3 : d~loglogN - ultra small world, λ>3 : d~logN.
- Optimal distance – strong disorder – Random Graphs and WS $\boldsymbol{l_{opt}} \sim \boldsymbol{N}^{1/3}$ Large World

scale free
$$\begin{cases} l_{opt} \sim N^{\frac{\lambda-3}{\lambda-1}} & for \quad \lambda > 3 \quad \Rightarrow \text{Large World} \\ l_{opt} \sim \log^{\lambda-1} N & for \quad 2 < \lambda < 3 \quad \Rightarrow \text{Small World} \end{cases}$$

- Transition between weak and strong disorder
- Scale Free networks (2<λ<3) are robust to random breakdown.
- Scale Free networks are vulnerable to attack on the highly connected nodes.
- Efficient immunization is possible without knowledge of topology, using Acquaintance Immunization.
- The critical exponents for scale-free directed and non-directed networks are different than those in exponential networks – different universality class!
- Large networks can have their connectivity distribution optimized for maximum robustness to random breakdown and/or intentional attack.