Fractals

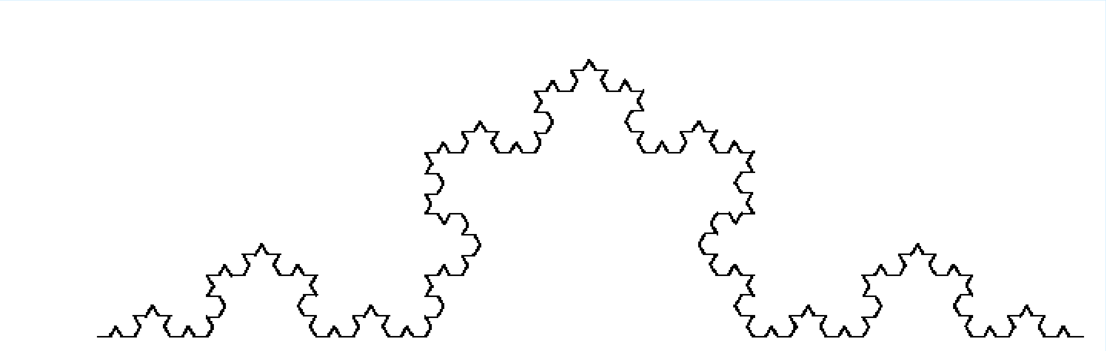
Fractal geometry describes Nature better than classical geometry.
Two types of fractals: deterministic and random.

Deterministic fractals

Ideal fractals having self-similarity.
Every small part of the picture when magnified properly, is the same as the whole picture.

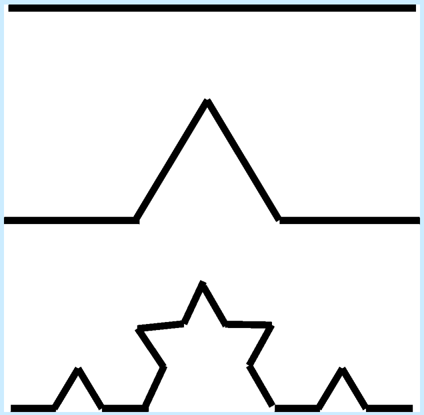
Self-similarity is a property, not a definition

To better understand fractals, we discuss several examples:



Koch curve

Building Koch curve



$n=0$

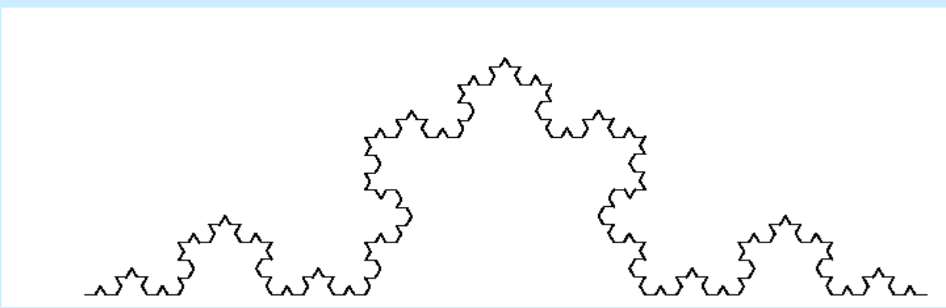
A section of unit length

$n=1$

Divide each section to 3 equal pieces and replace the middle one by two pieces like a tent

$n=2$

The same is done for all 4 sections



$n=\infty$

This is a mathematical fractal

In physics we continue until $n_{\max}$. We have a fractal for length scales $1/3^{n_{\max}} < x < 1$

Koch curve properties:

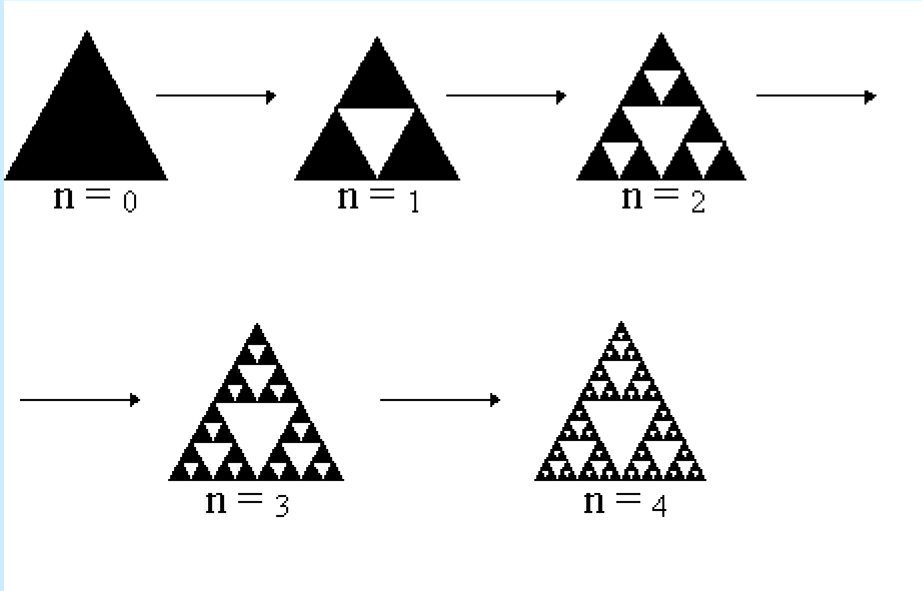
(a) $\left(\frac{4}{3}\right)^n = Length \to \infty \quad for \quad n = \infty$. But contains in a finite space. No derivative.

(b) Self-similarity – scale invariance

(c) No characteristic scale

Sierpinski gasket is perhaps the most popular fractal.

Generation of Sierpinski gasket

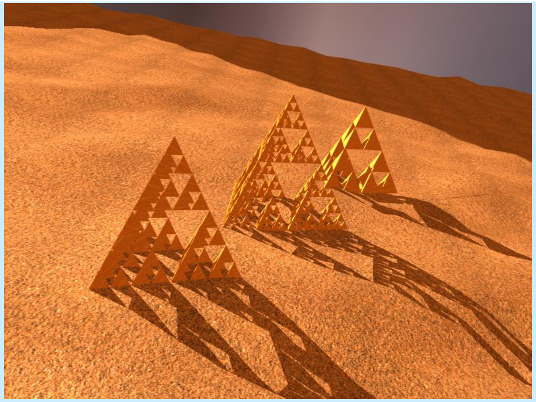


(1) divide an equilateral triangle into 4 equal triangles
(2) take out the central one
(3) repeat this for every triangle

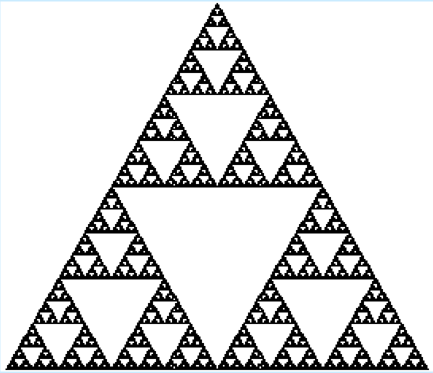
No translation symmetry
Scale invariance symmetry

Internal perimeter: $\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\ldots\to\infty$

Area: $S_0, \frac{3}{4}S_0, \left(\frac{3}{4}\right)^2 S_0 \ldots\to 0$

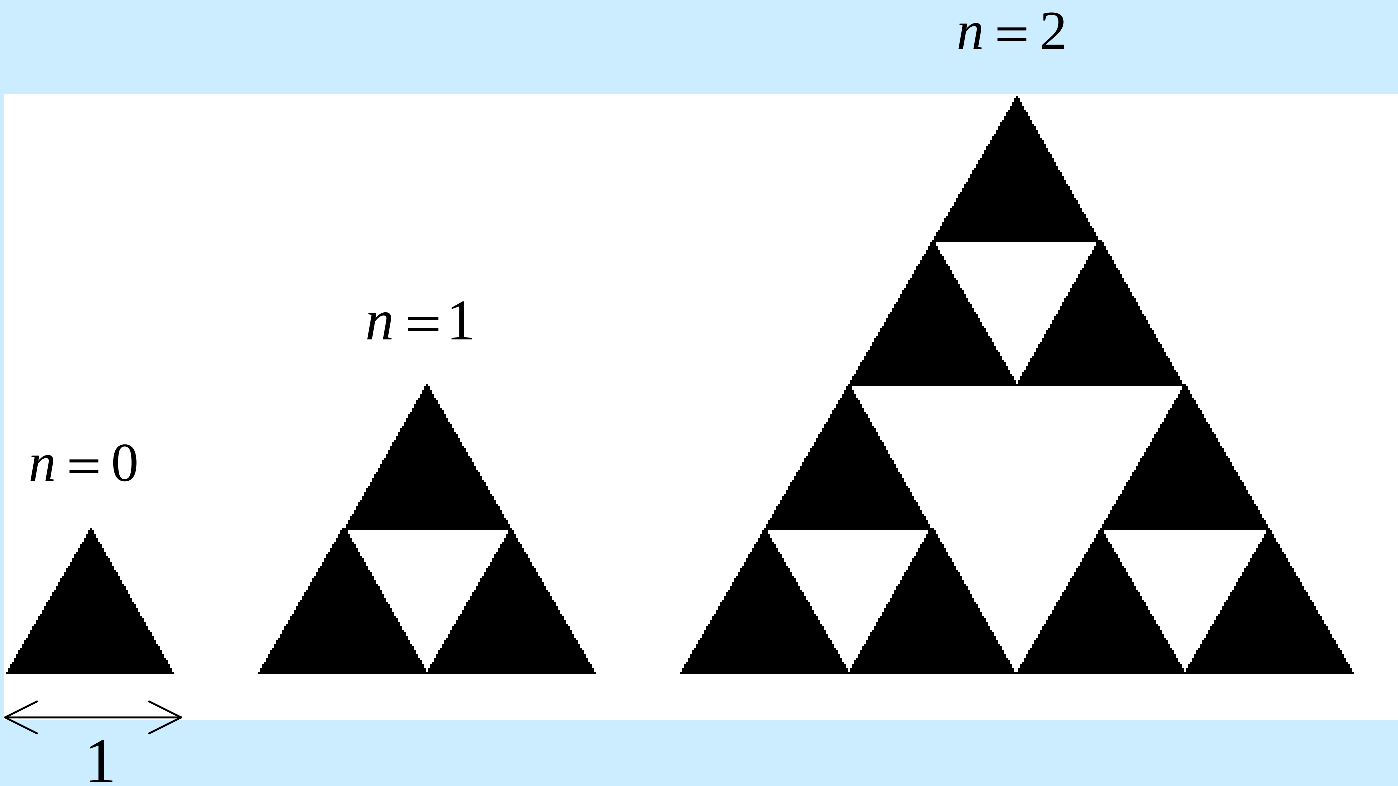


3D Sierpinski gasket



2D Sierpinski gasket

Sierpisnki gasket with lower cut off

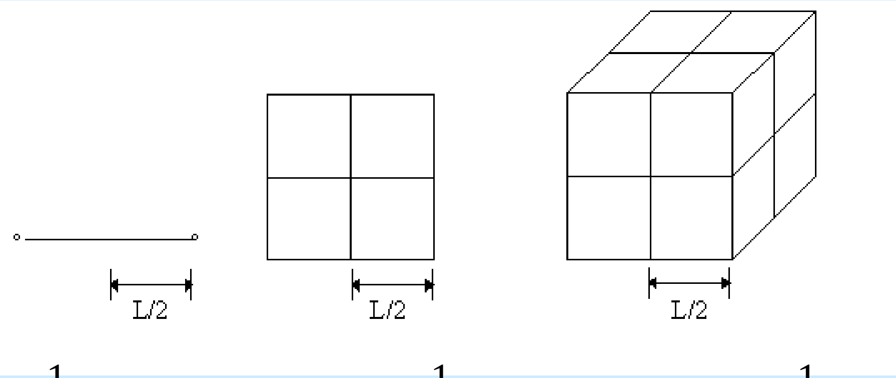


This is a fractal for $1 < x < 3^{n_{\max}}$

Fractal dimension

How to quantify fractals ?
Generalization of dimension to non-integer dimensions – fractal dimension (B.B. Mandelbrot, 1977)

Definition of dimension



$$M(L/2)=\frac{1}{2}M(L) \qquad M(L/2)=\frac{1}{4}M(L) \qquad M(L/2)=\frac{1}{8}M(L)$$

$$d=1 \qquad d=2 \qquad d=3$$

- Take a line section of length L, divide into two, we get: $M\left(\frac{1}{2}L\right)=\frac{1}{2}M(L)$
- Take a square of length L, divide L by 2 we get: $M\left(\frac{1}{2}L\right)=\frac{1}{4}M(L)=\frac{1}{2^2}M(L)$
- Take a qube of length L, divide L by 2 we get: $M\left(\frac{1}{2}L\right)=\frac{1}{8}M(L)=\frac{1}{2^3}M(L)$

In general

$$M(bL)=b^d M(L)$$

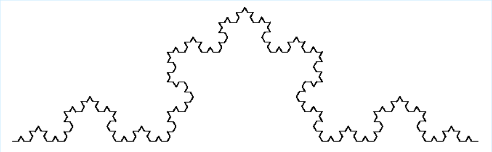
The exponent d defines the dimension of system

Solution: $M(L)=AL^d$ where A is a constant

Definition of fractal dimension $M(bL) = b^{d_f} M(L)$

generalization to non-integer dimension d_f

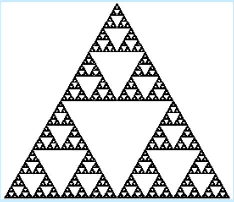
Solution: $M(L) = AL^{d_f}$



Example: Koch curve

$$M\left(\frac{1}{3}L\right) = \frac{1}{4}M(L) = \left(\frac{1}{3}\right)^{d_f} M(L) \Rightarrow \left(\frac{1}{3}\right)^{d_f} = \frac{1}{4} \quad or \quad d_f = \frac{\log 4}{\log 3} \approx 1.262$$

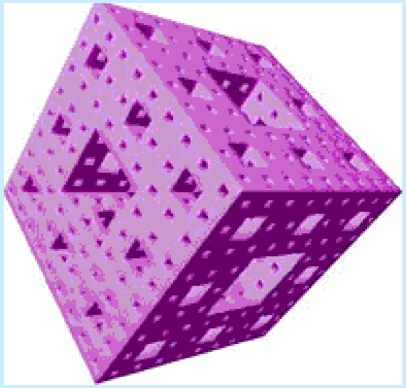
d_f - non integer – between 1 and 2 dimensions. Koch curve is not a line (d=) but doesn't fill a plane (d=).



Example: Sierpinski gasket

$$M\left(\frac{1}{2}L\right) = \frac{1}{3}M(L) = \left(\frac{1}{2}\right)^{d_f} M(L) \Rightarrow \left(\frac{1}{2}\right)^{d_f} = \frac{1}{3} \quad or \quad d_f = \frac{\log 3}{\log 2} \approx 1.585$$

Non integer dimension between 1 and 2 dimensions.



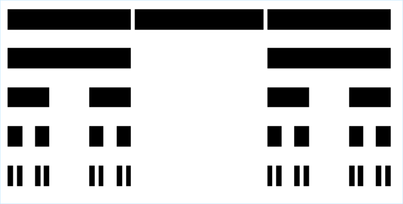
Example: Sierpinski sponge:

$$M\left(\frac{1}{3}L\right)=\frac{1}{20}M(L)=\left(\frac{1}{3}\right)^{d_f}M(L)\Rightarrow\left(\frac{1}{3}\right)^{d_f}=\frac{1}{20}\quad or\quad d_f=\frac{\log 20}{\log 3}\approx 2.727$$

Here the fractal dimension is between 2 and 3.

Are there fractals with $d_f<1$?

Example: Cantor set



A section of unit size.
Divide into 3 equal sections
and remove the central one.
Repeat it for every left section.
For $n\to\infty$ we get a fractal set
of points.

$$M\left(\frac{1}{3}L\right)=\frac{1}{2}M(L)=\left(\frac{1}{3}\right)^{d_f}M(L)\Rightarrow\left(\frac{1}{3}\right)^{d_f}=\frac{1}{2}\quad or\quad d_f=\frac{\log 2}{\log 3}\approx 0.631$$

Relation between fractals and chaos

Cantor set is related to chaos. In chaotic systems we have strange fractal attractors.

Logistic map: $\boxed{x_{t+1} = l\, x_t (1 - x_t), \quad t = 1,2,3\ldots}$

- Nonlinear dynamical equation

Model for the dynamics of biological populations:

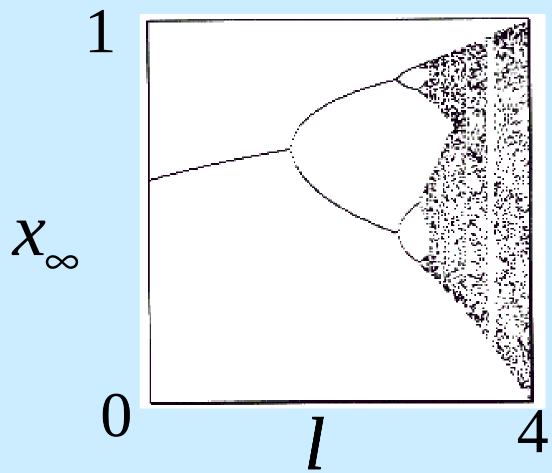
1st term – exponential growth: $x_{t+1} = l\, x_t$ (enough food, no diseases, no predators)
2nd term – decay $-l\, x_t^2$

For $0 \le l \le 4$ and $0 < x_0 < 1$: follows $0 < x_t < 1$.
The dynamics of x_t (for large t) depends on l .
For $l < l_1 = 3$: a single stable fixed point x_t approaches to same value for any x_0 .
At $l_1 = 3$: the fixed point bifurcates (two stable fixed points).
For large t the trajectories move periodically between two values with a period of $\Delta t = 2$.
For example, for $l = 3.1$ after about 200 iterations x_t obtains the values . … and . …

For $l = l_2 = 1+\sqrt{6} \simeq 3.449$: each of the two fixed points bifurcates again to two new fixed points.

The trajectories have a period of $\Delta t = 4$ along those 4 points.

For higher values of l new bifurcations occur at l_n with a period of $\Delta t = 2^n$ between l_n and l_{n+1}.



For large n the difference between l_{n+1} and l_n becomes smaller according to:

$$l_{n+1} - l_n = (l_n - l_{n-1})/d$$

with $d \cong 4.6692$ called Feigenbaum constant who found that d is universal for all quadratic maps.

For $l_\infty \cong 3.5699456\ldots$ the period is infinite and x_i moves chaotically between the infinite fixed points.

The set of these infinite points is called strange attractor and represent a Cantor set with fractal dimension $d_f \cong 0.538$.

Above l_∞ more complex dynamics occurs which is beyond the scope of this course.