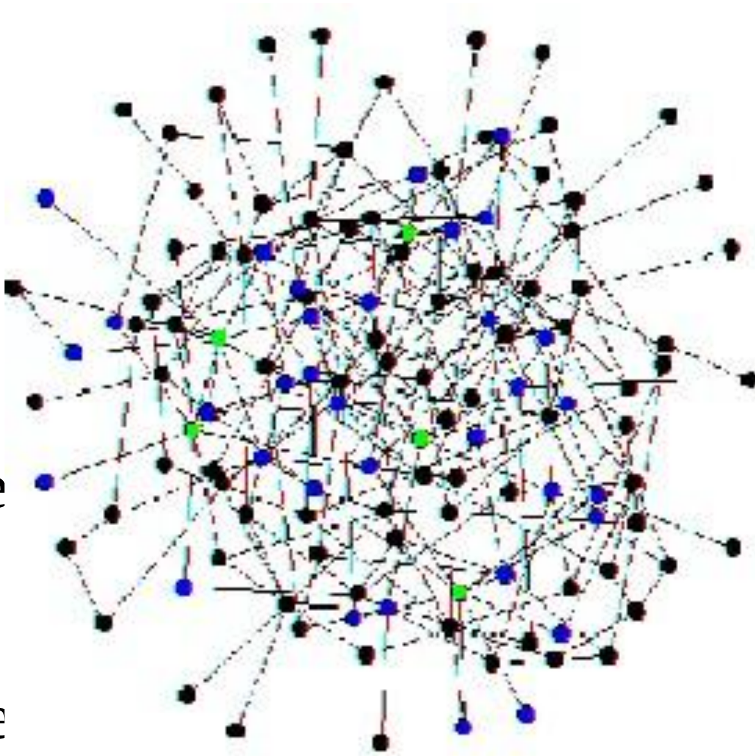
# Complex Networks

- Network is a structure of N nodes and 2M links (or M edges)
- Called also graph – in Mathematics
- Many examples of networks



Internet: nodes represent computers

links the connecting cables

Social network: nodes represent people

links their relations

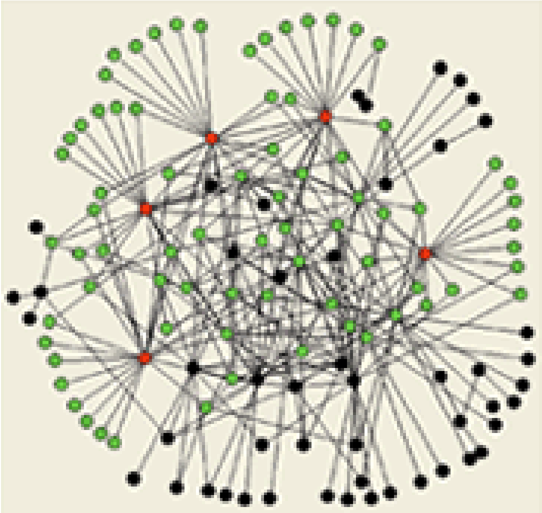
Cellular network: nodes represent mole

links their interactions

# Networks as a tool for Complex systems

- Complex systems are usually composed of many interacting entities.

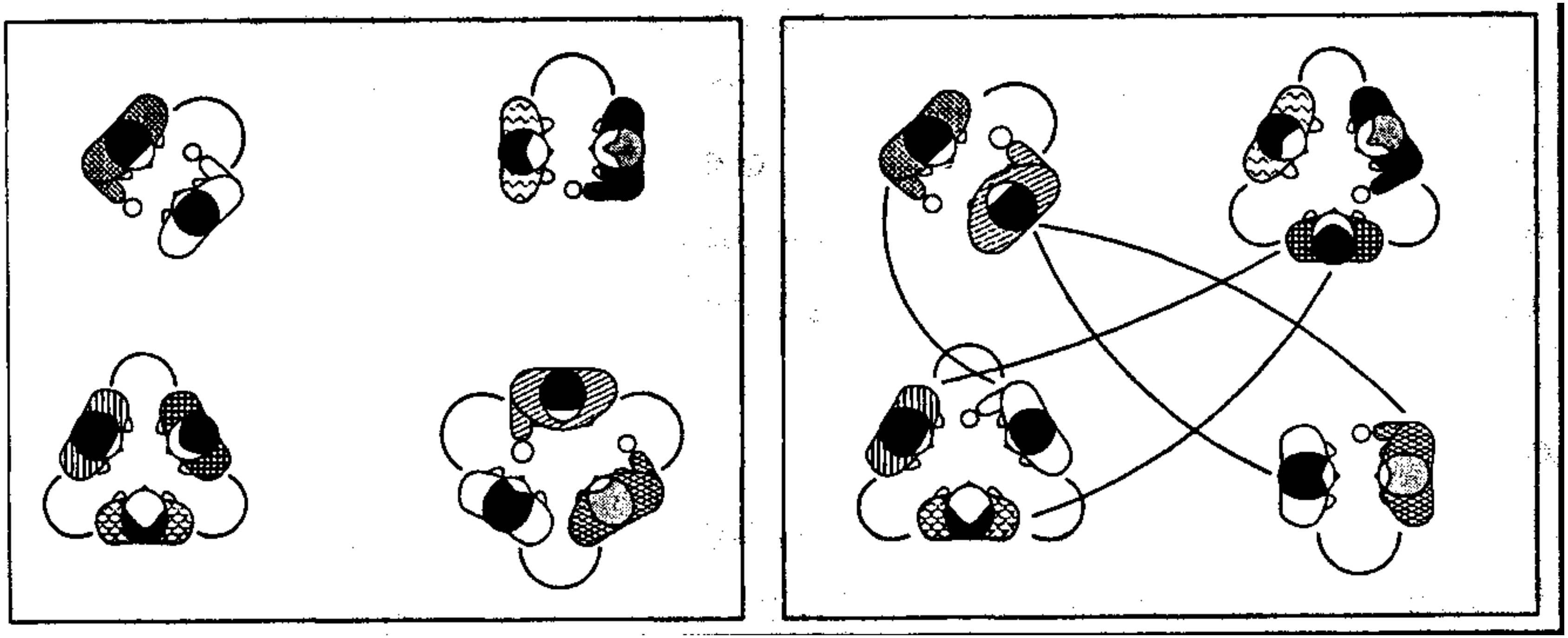
- Can be well represented by networks: nodes represent the entities and links their interactions.



- Examples: biological systems, social systems, climate, earthquakes, epidemics, transport, economic, etc.

- Weighted networks each link (node) has a weight determining the strength or cost of the link (node).

# Social Networks- Stanley Milgram (1967)

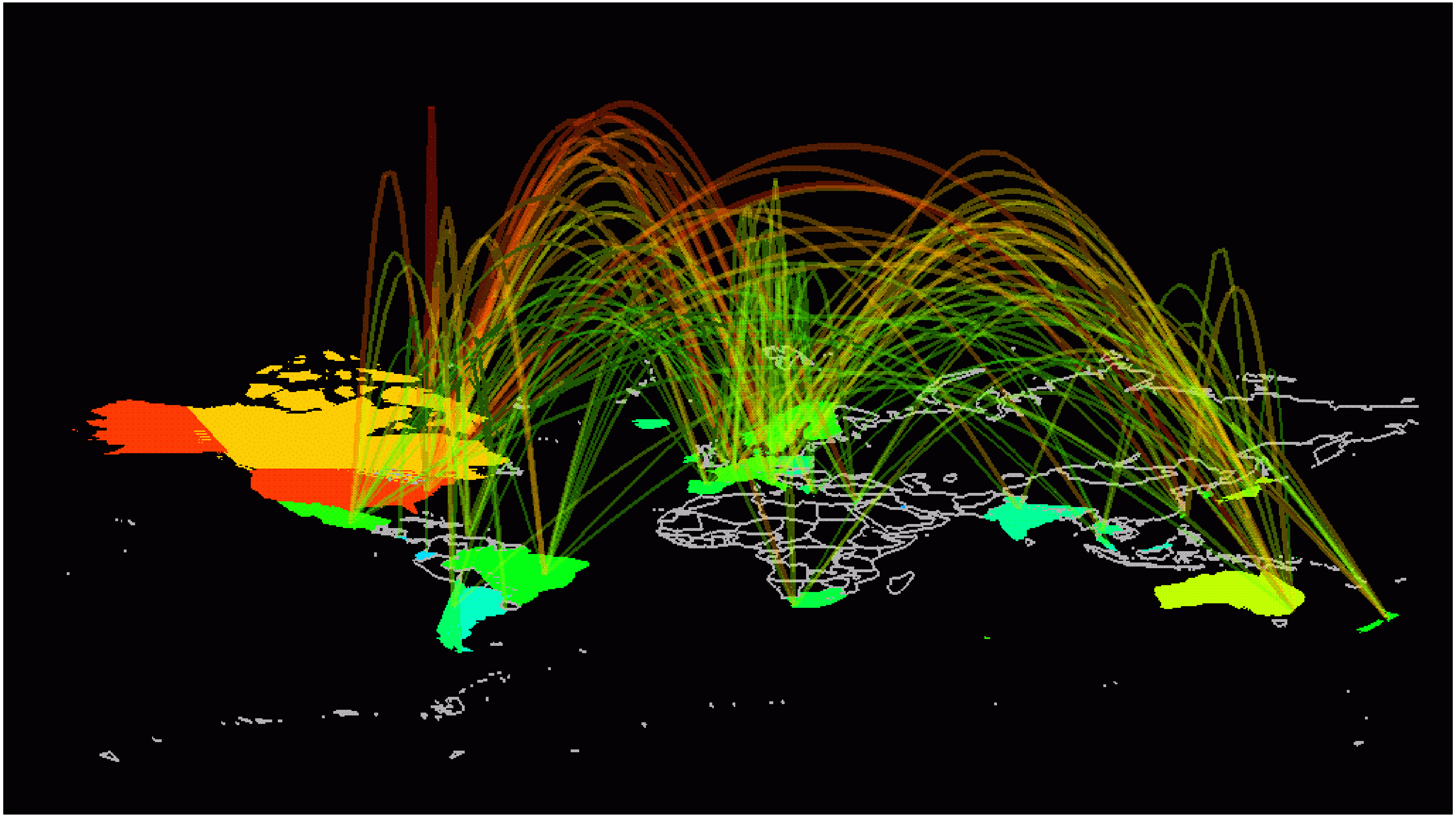


**Nodes**: individuals

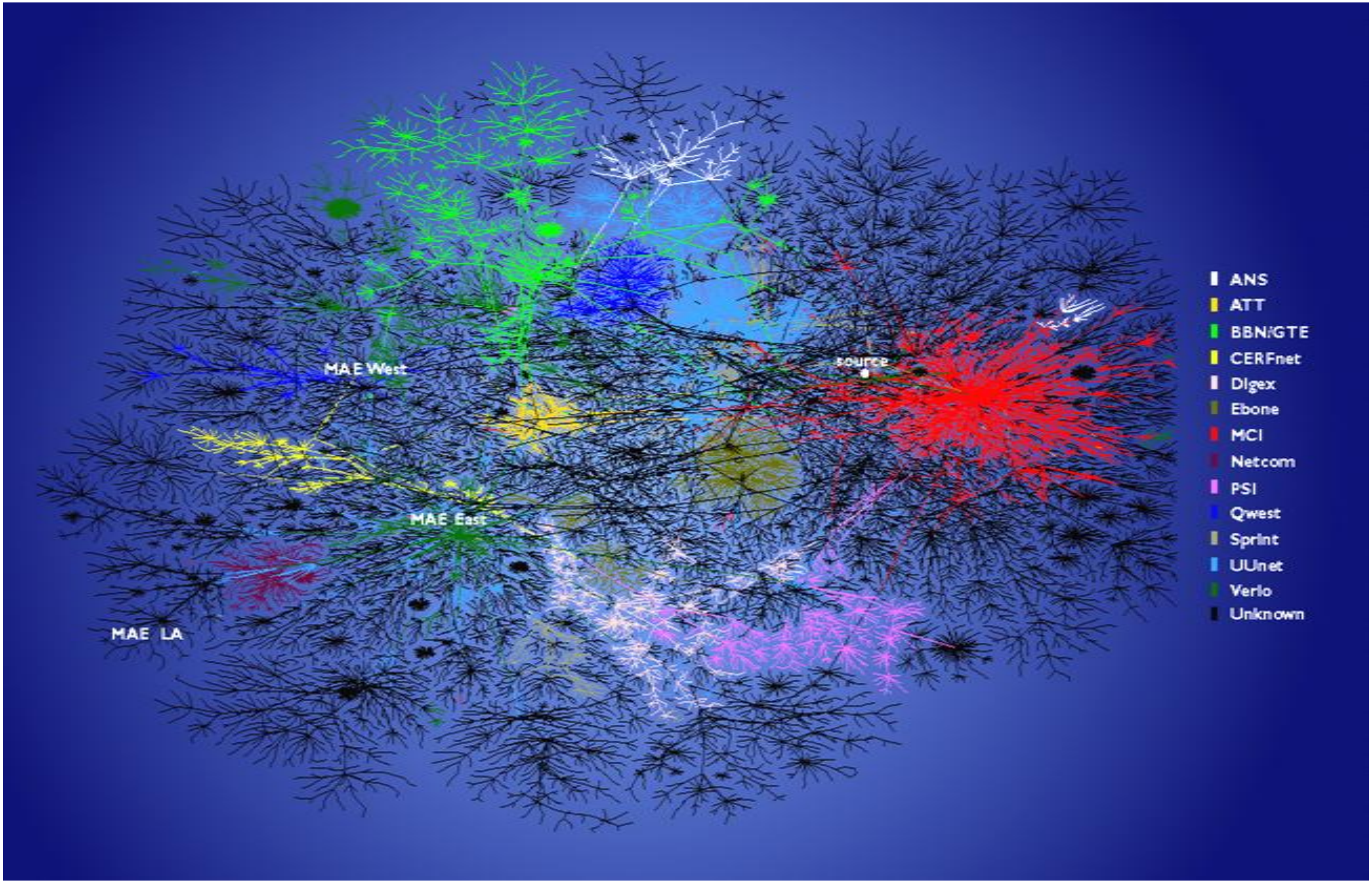
**Links**: social relationship (family/work/friendship/etc.)

John Guare (1992)

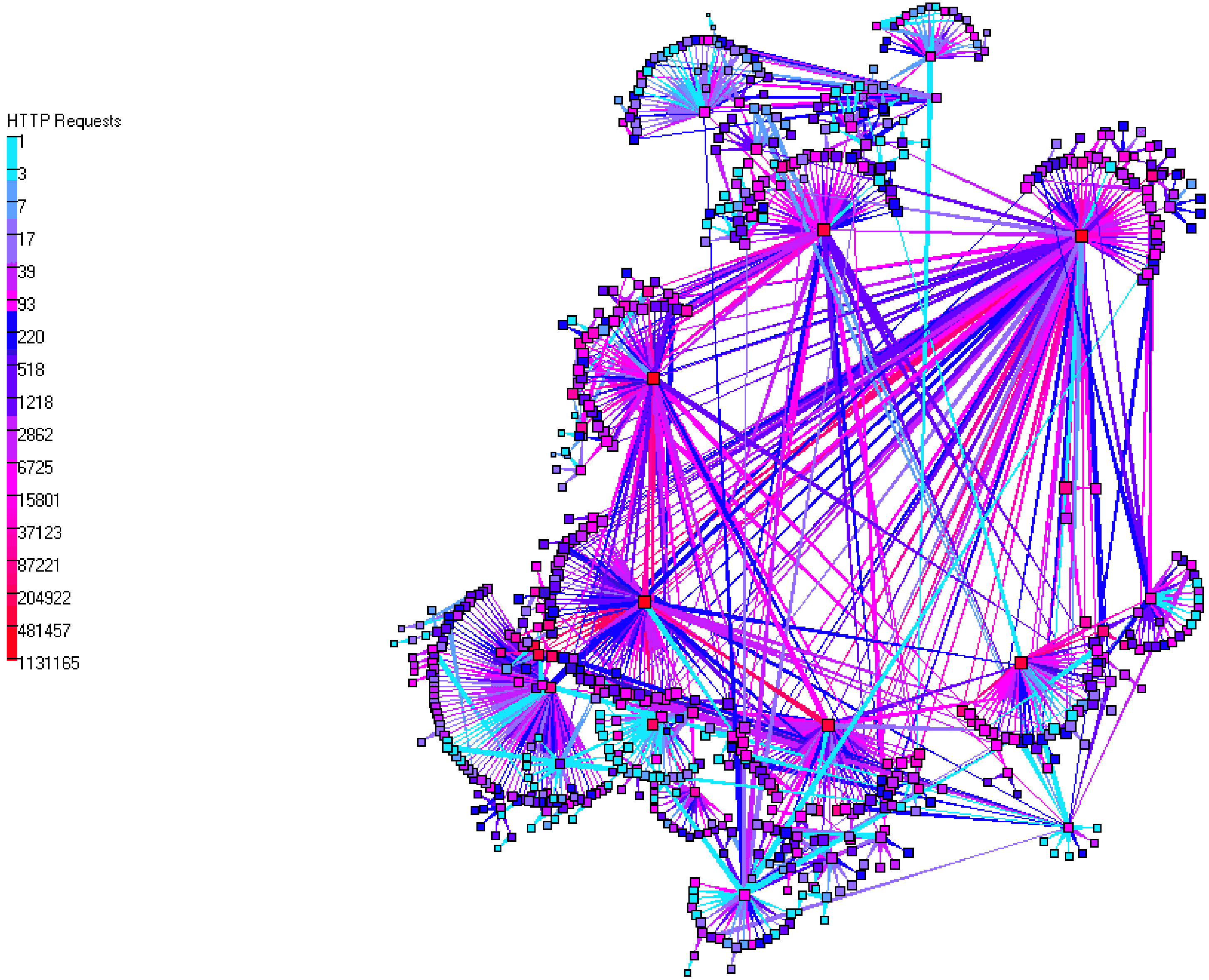
**Six Degrees of Separation**



Map showing the world-wide internet traffic




A snapshot of Internet connectivity.




Hierarchical topology of the international web cache

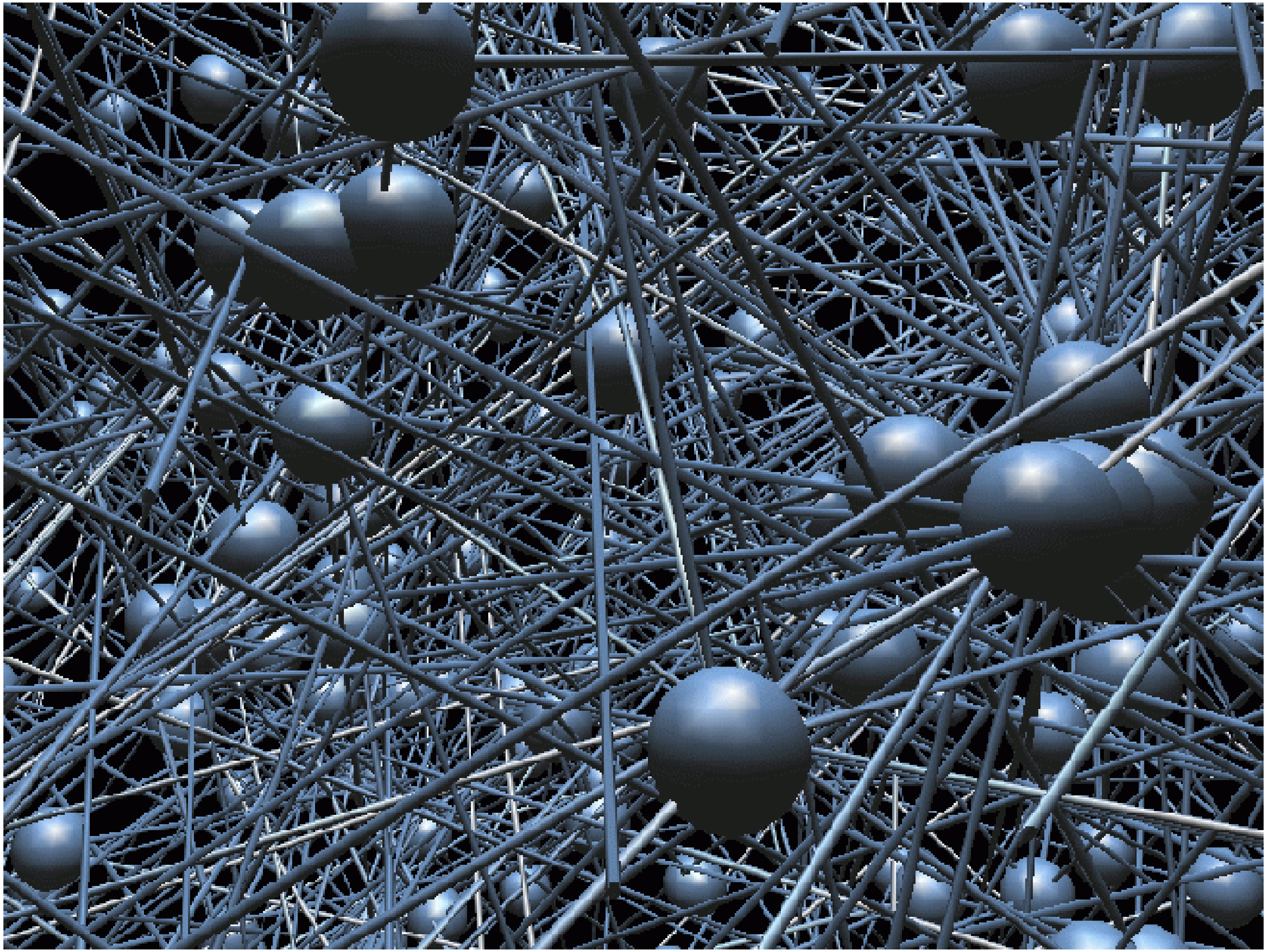
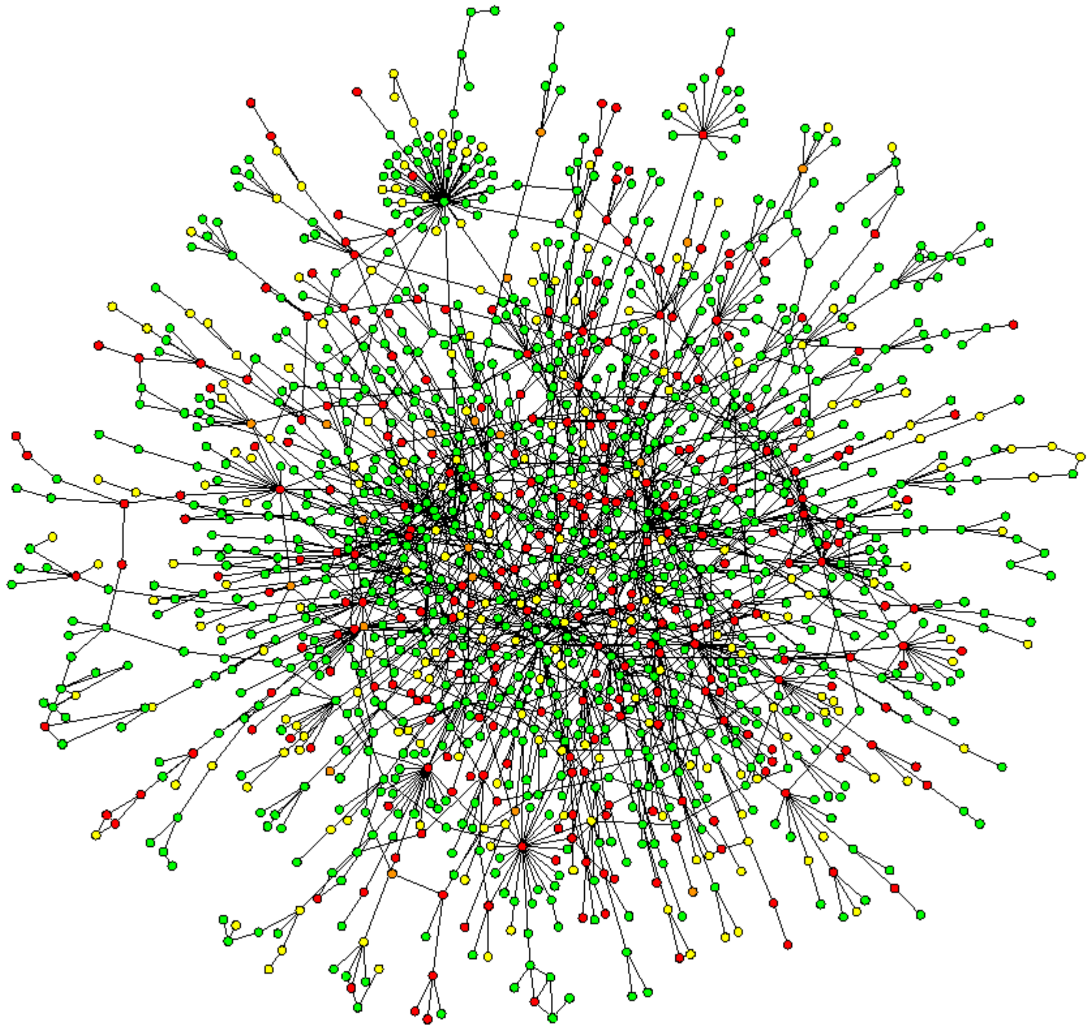
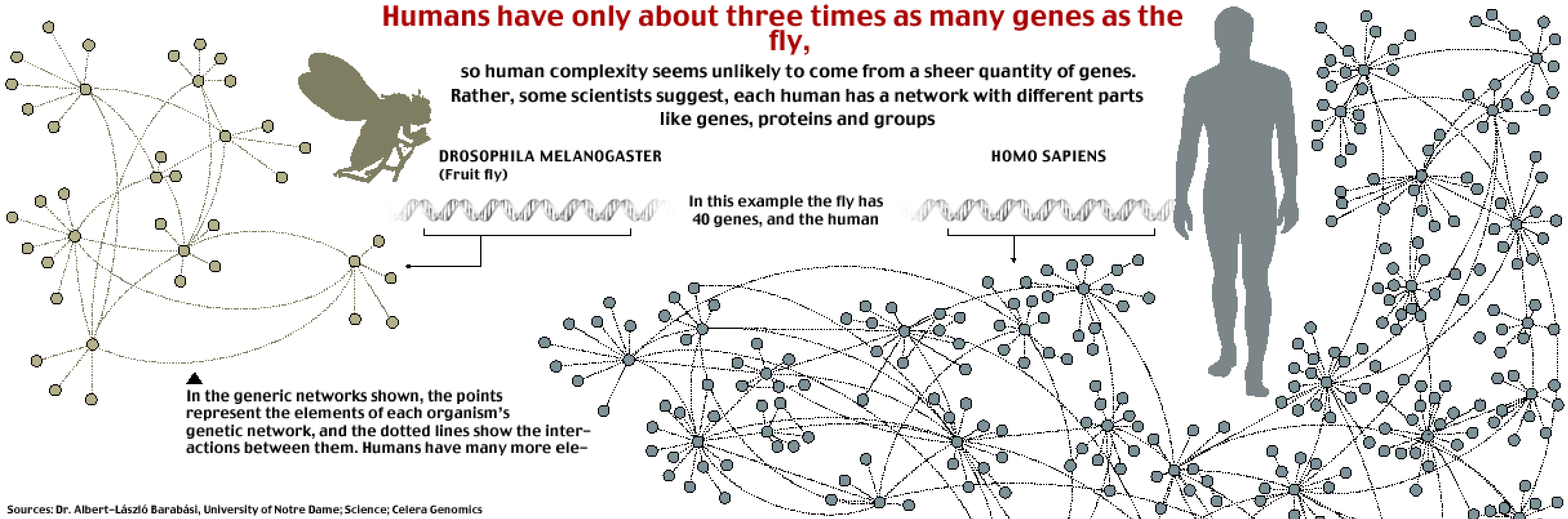
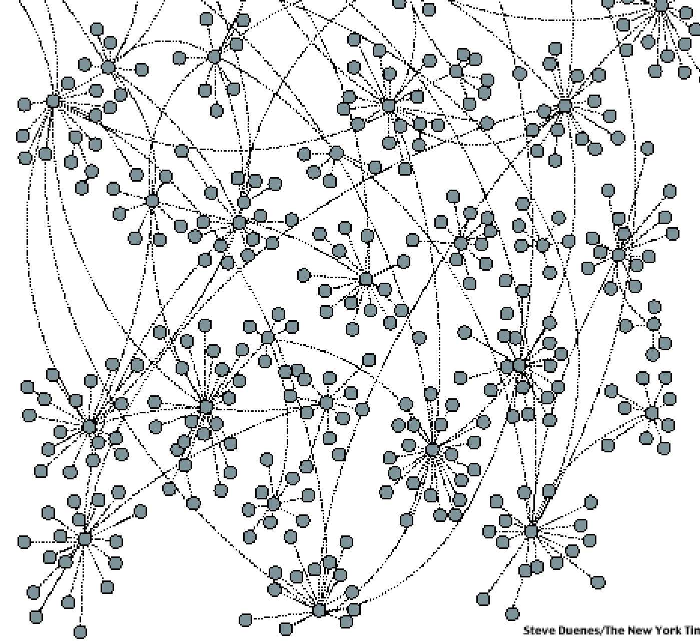


Image of Social links in Canberra, Australia



Network of protein-protein interactions. The color of a node signifies the phenotypic effect of removing the corresponding protein (red, lethal; green, non-lethal; orange, slow growth; yellow, unknown).




# Complex systems

Made of

many non-identical **elements**

connected by diverse **interactions**.

NETWORK

# Network Properties

❖ Degree distribution P(k) -- k- degree of a node

❖ Diameter or distance – Average distance between nodes--d

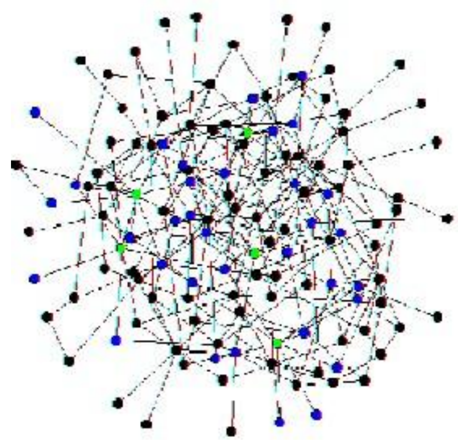
❖ Clustering Coefficient c(k)= $\frac{\text{no. of links between k neighbors}}{k(k-1)/2}$

How many of my friends are also friends?

❖ Centrality or Betweeneness -- b

Number of times a bond or a node is relatively used for the shortest path

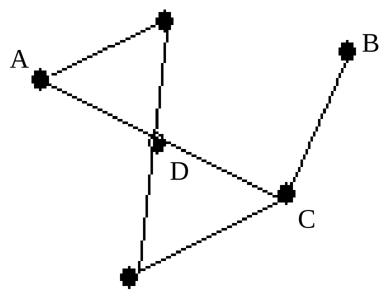
❖ Critical Threshold: The concentration of nodes that are removed and the network collapses



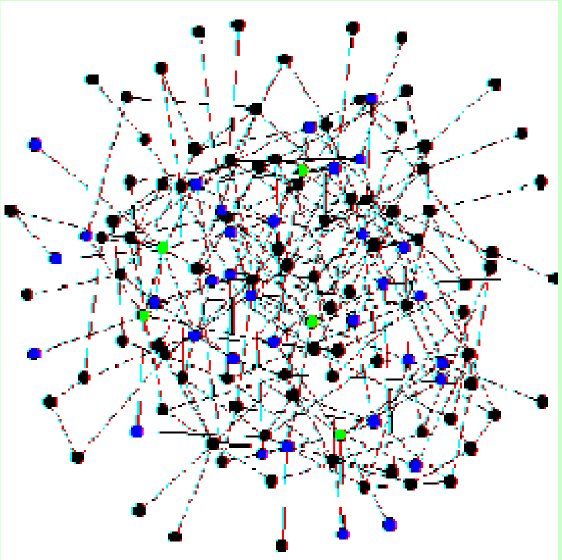
$$d(AB) = 3$$

$$c(D) = \frac{2}{6} = \frac{1}{3}$$

$$b(BC) = \frac{N-1}{N-1} = 1$$



# Random Graph Theory



- **Developed in the 1960's by Erdos and Renyi.** **(Publications of the Mathematical Institute of the Hungarian Academy of Sciences, 1960).**

- **Discusses the ensemble of graphs with N vertices and M edges (2M links).**
- **Distribution of connectivity per vertex is Poissonian (exponential), where $k$ is the number of links :**

$$P(k) = e^{-c}\frac{c^k}{k!}, \qquad c = \langle k \rangle = \frac{2M}{N}$$

- **Distance d=log N -- SMALL WORLD**

# More Results

- **Phase transition at average connectivity, $\langle k \rangle = 1$:**

  $\langle k \rangle < 1$ **No spanning cluster (giant component) of order** $\log N$

  $\langle k \rangle > 1$ **A spanning cluster exists (unique) of order** $N$

  $\langle k \rangle = 1$ **The largest cluster is of order** $N^{2/3}$

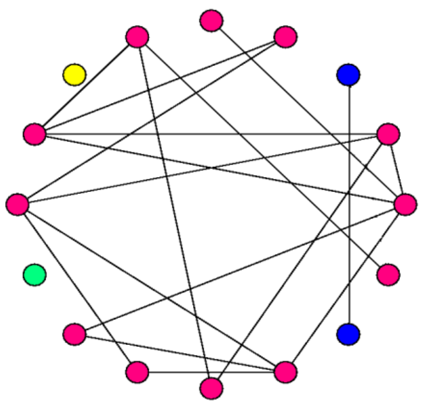
- **Behavior of the spanning cluster size near the transition is linear:**

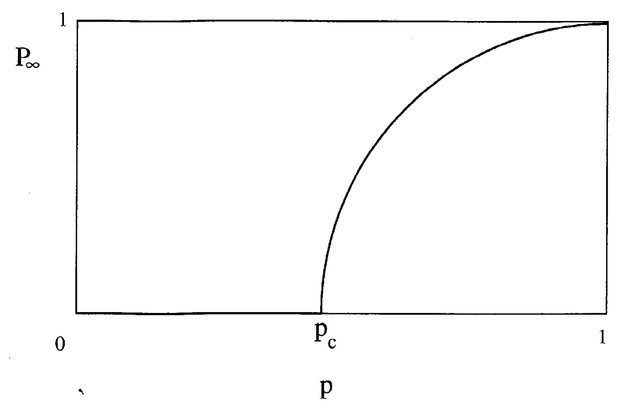
  $P_\infty \propto (p - p_c)^\beta$, $\beta = 1$, **where** $p$ **is the probability of a site to exist,**

  $p_c = 1/\langle k \rangle$

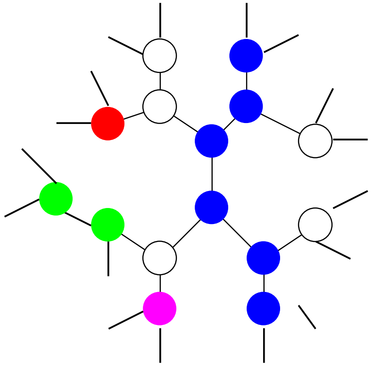
- **Size of the spanning cluster is determined by the self-consistent equation:**

  $P_\infty = 1 - e^{-\langle k \rangle P_\infty}$





# Percolation on a Cayley Tree



- **Contains no loops**

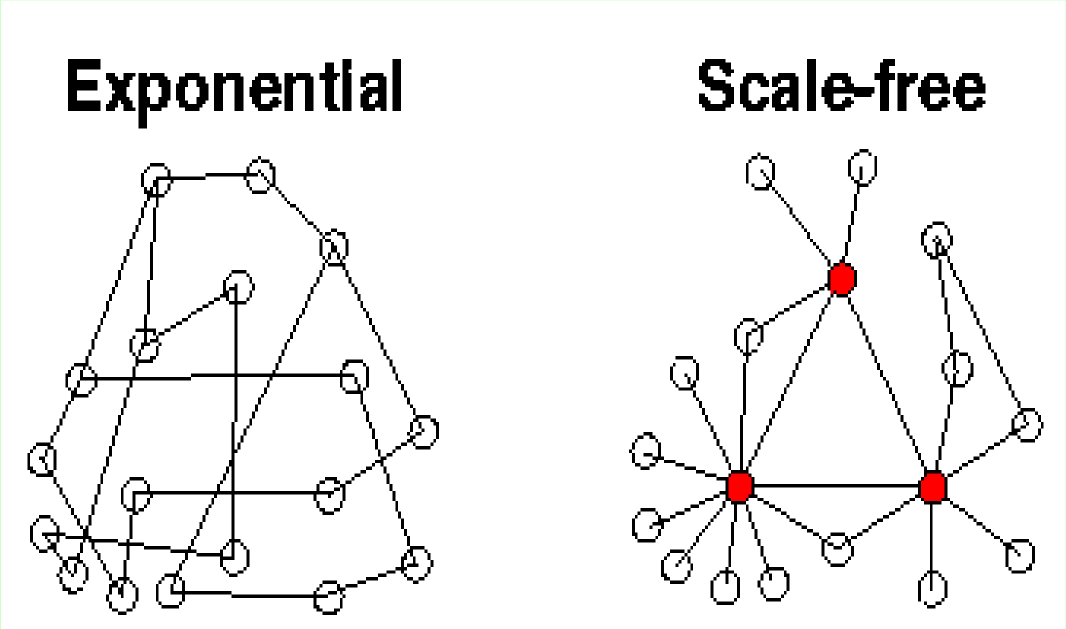
- **Connectivity of each node is fixed ($z$ connections)**

- **Critical threshold:** $$p_c = \frac{1}{z-1}$$

- **Behavior of the spanning cluster size near the transition is linear:**

$$P_\infty \propto (p_c - p)^\beta, \quad \beta = 1$$

# In Real World - Many Networks are non-Poissonian

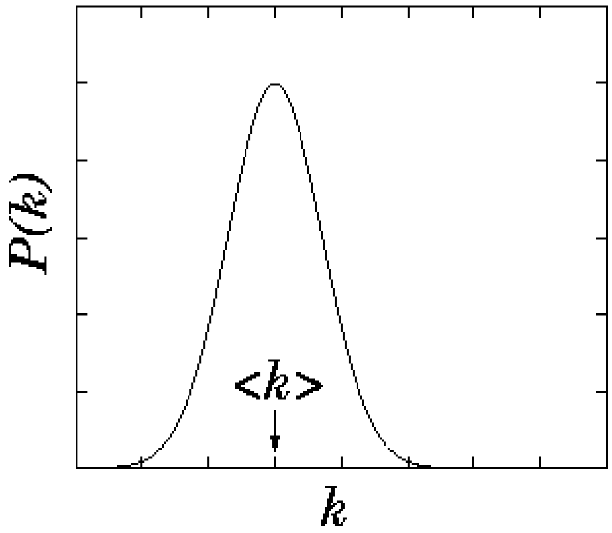



$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

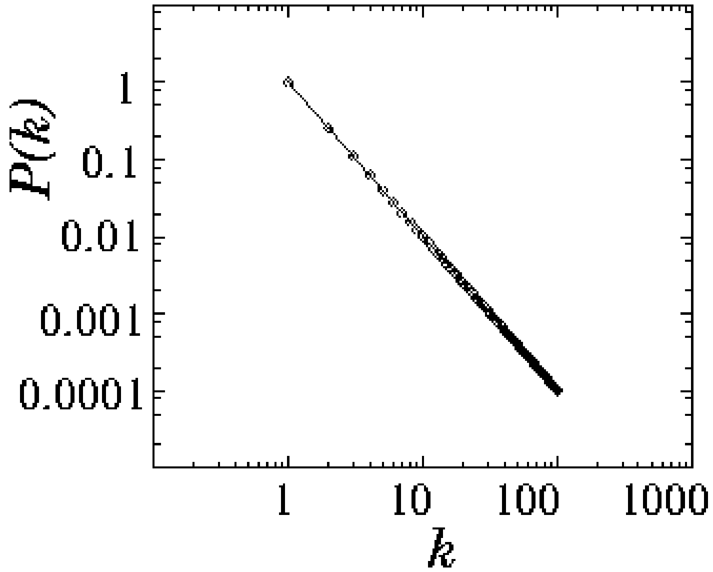
$$P(k) = \begin{cases} ck^{-\lambda} & m \le k \le K \\ 0 & otherwise \end{cases}$$

# New Type of Networks

Poisson distribution

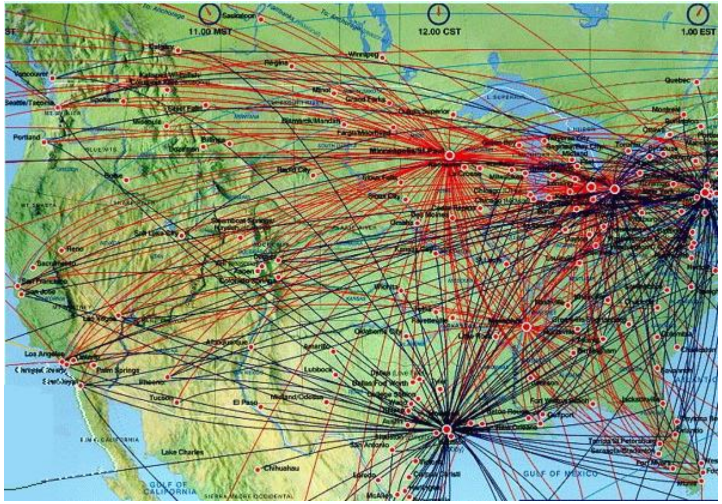



Power-law distribution



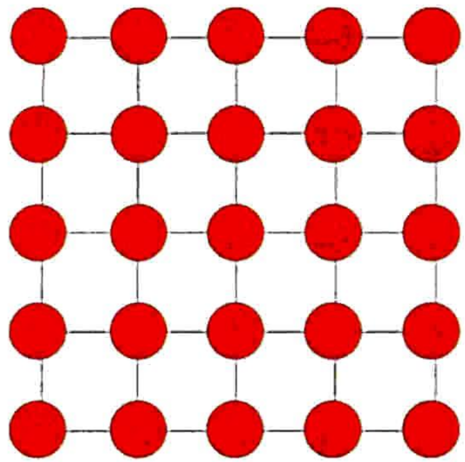


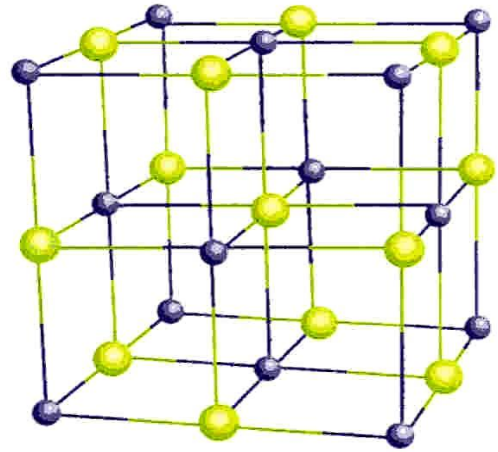
Exponential Network

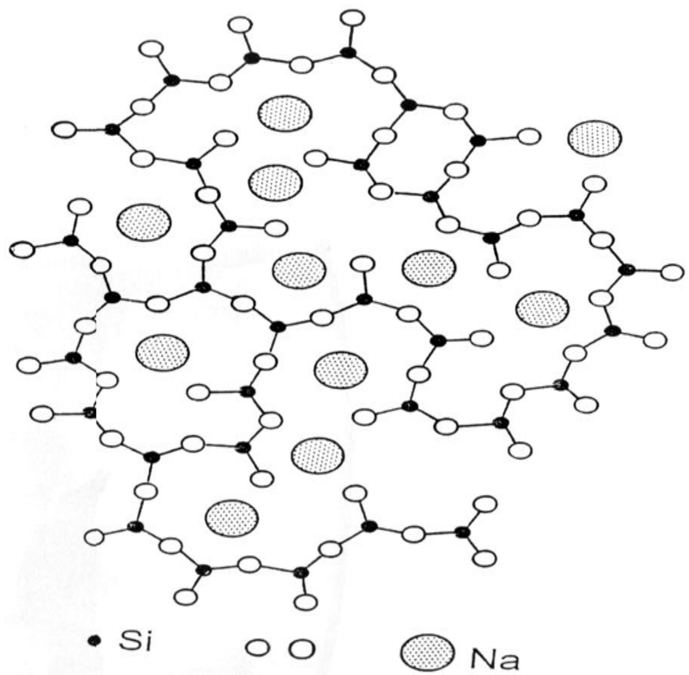


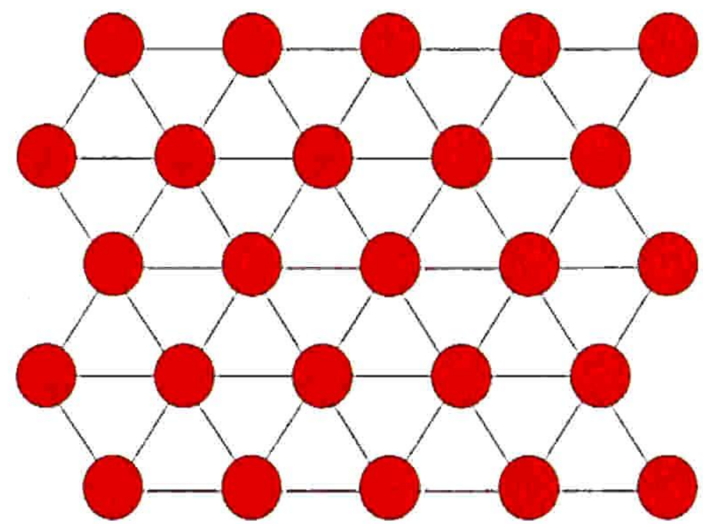
Scale-free Network

# Networks in Physics



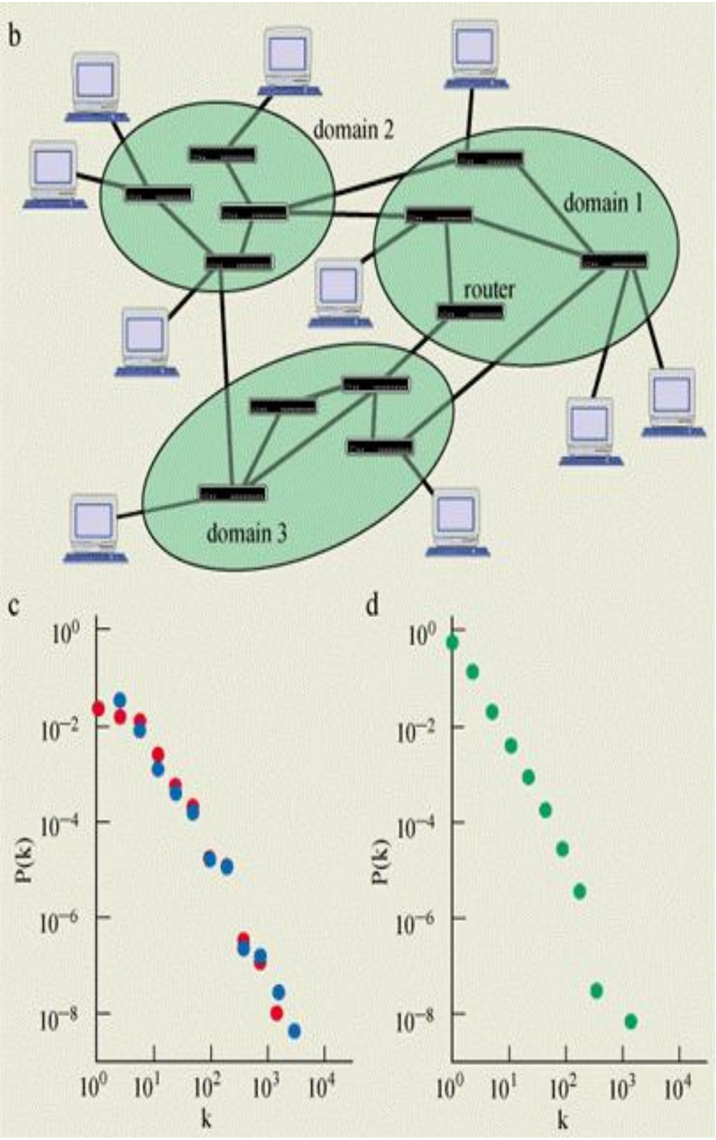


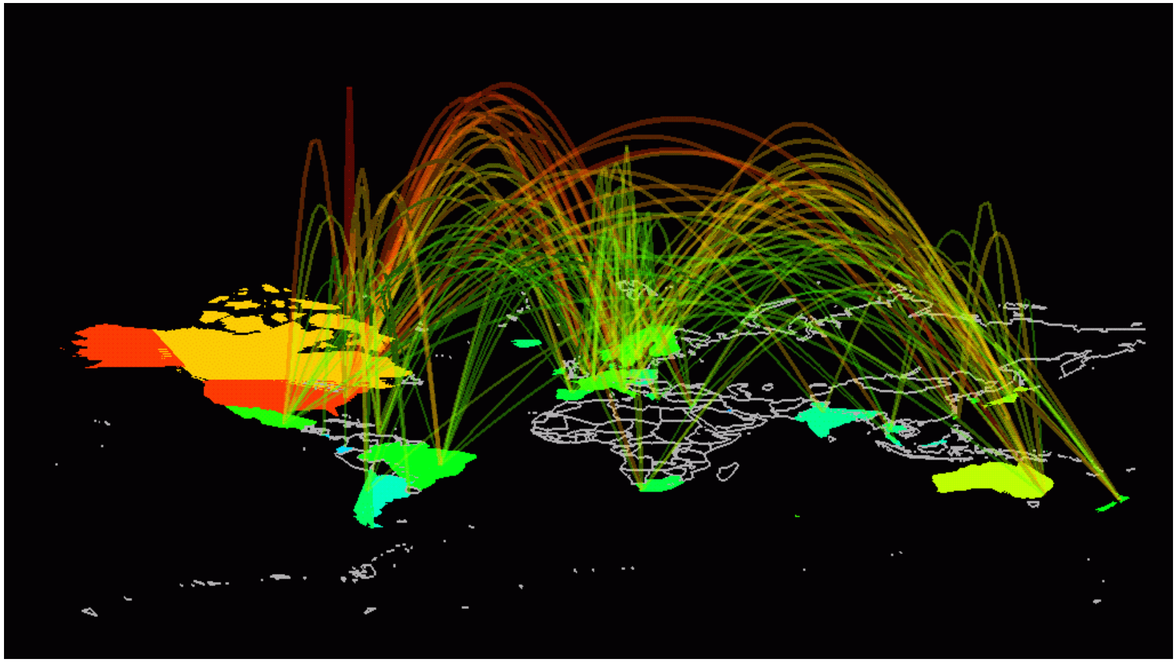


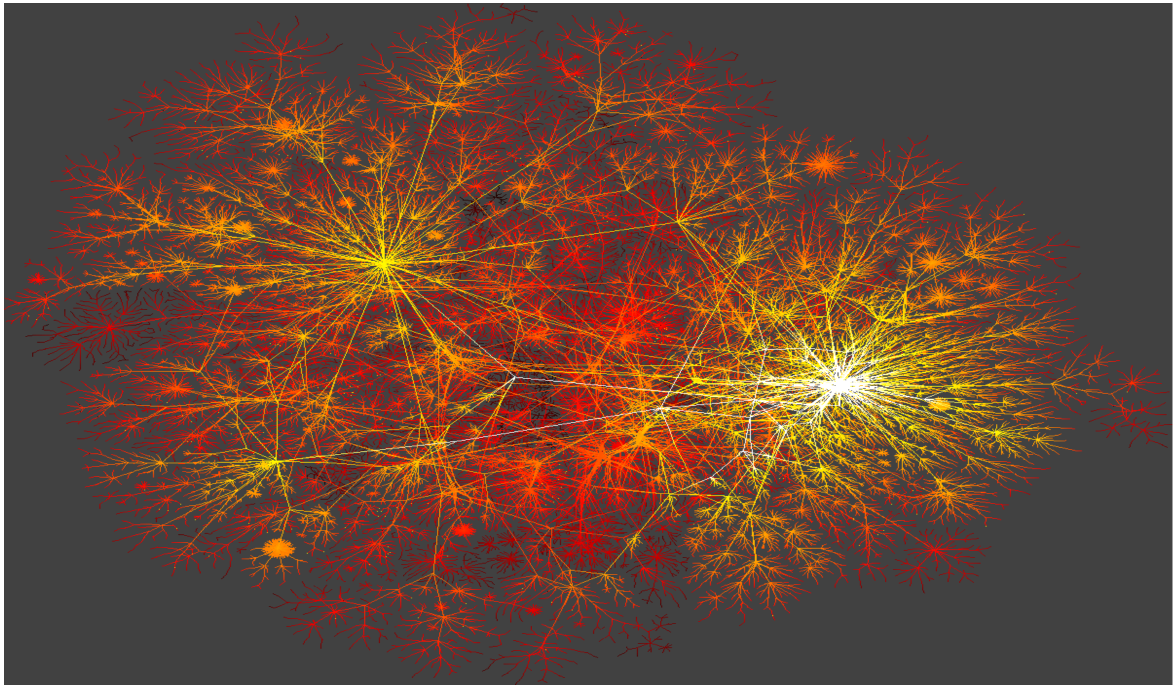



# Internet Network

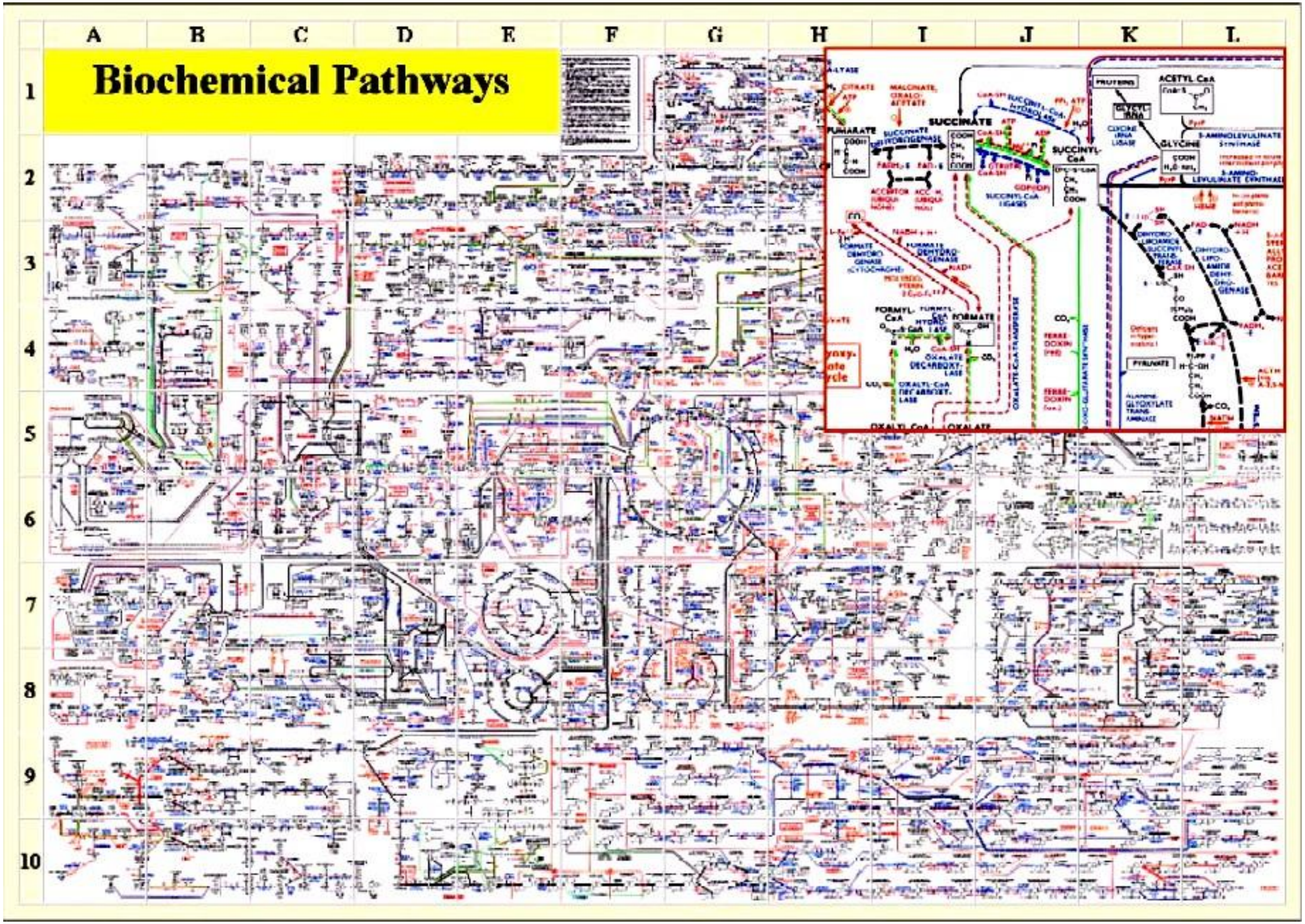
Faloutsos et. al., SIGCOMM '99





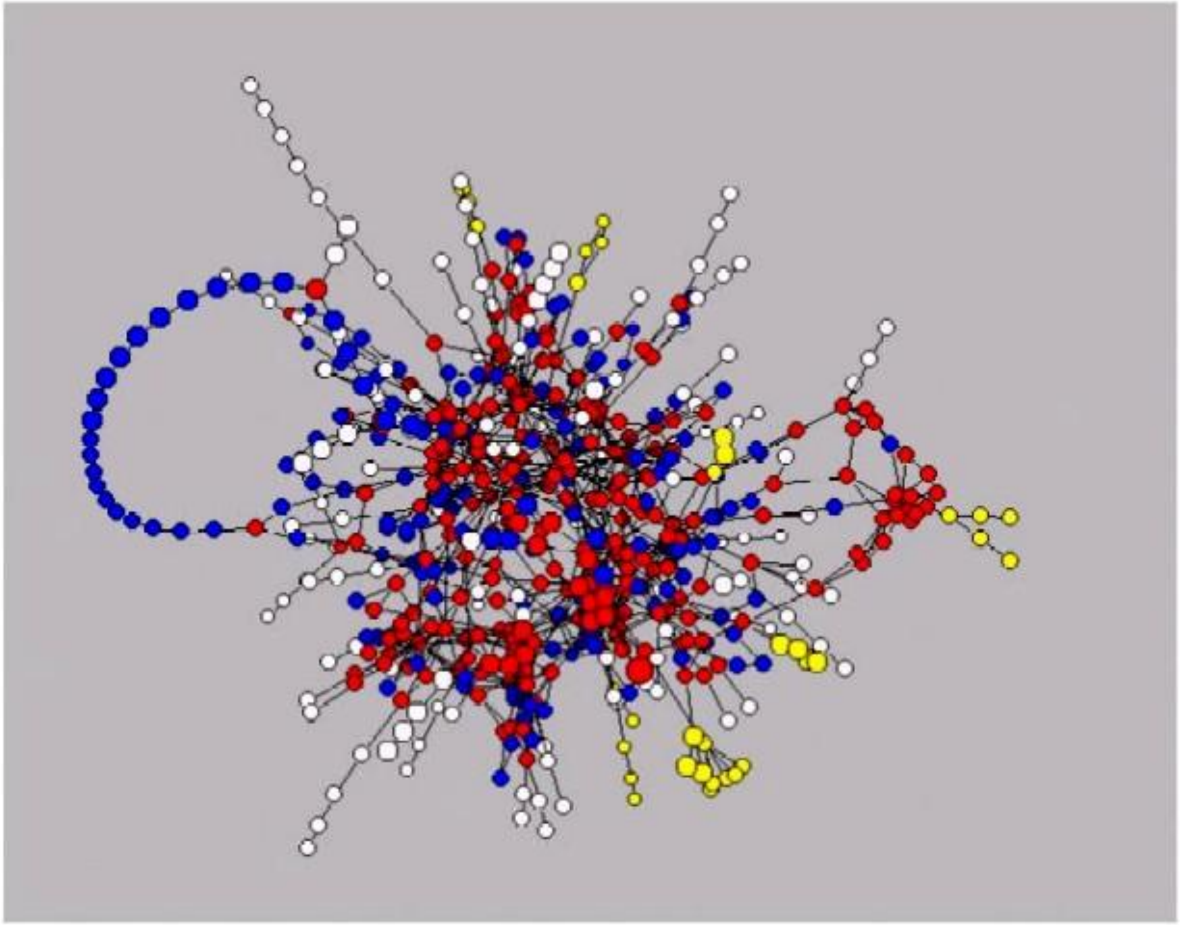
# Metabolic network



# Metabolic Network

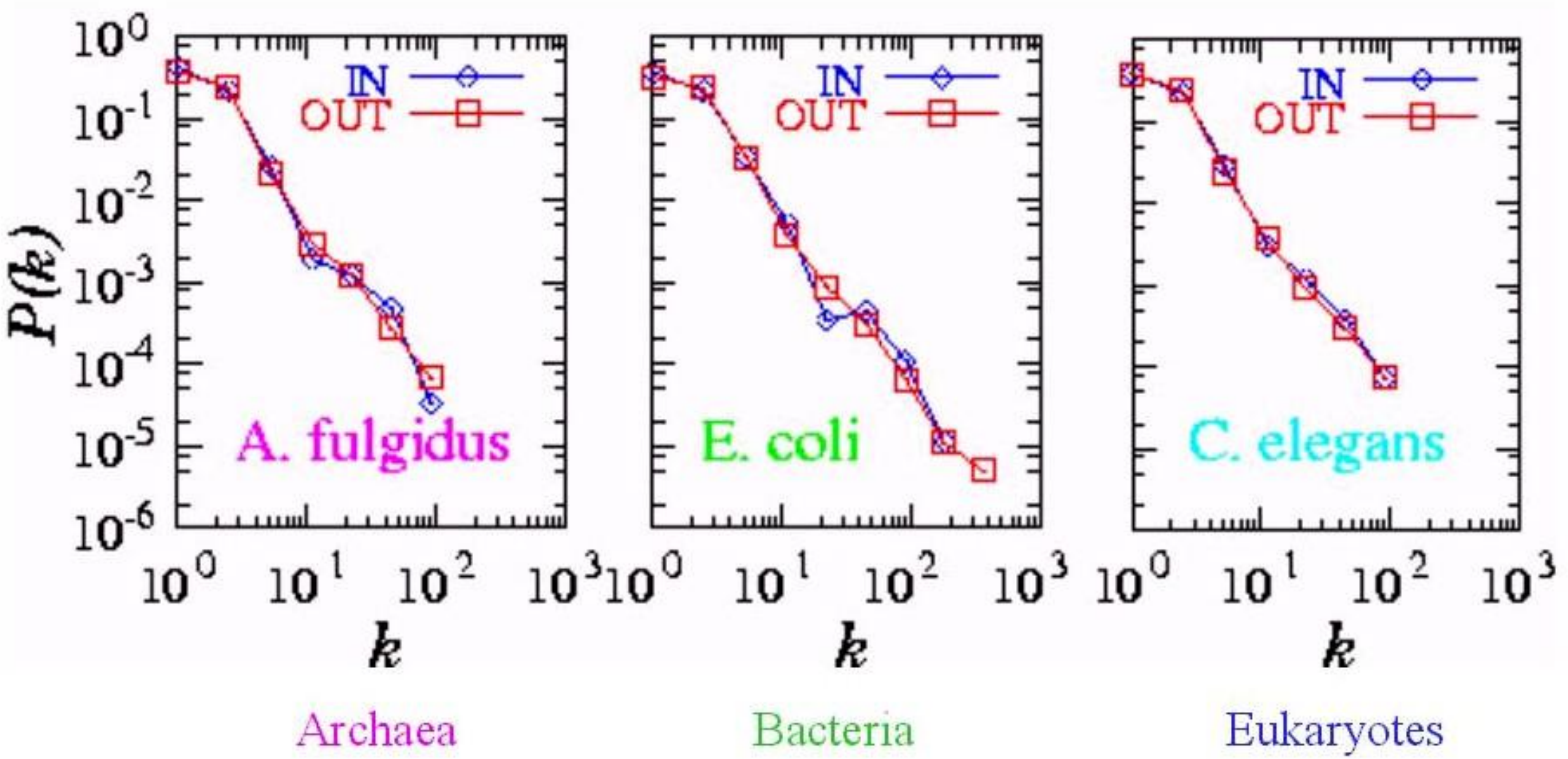
**Nodes**: chemicals (substrates)

**Links**: bio-chemical reactions



Jeong et all Nature 2000

# Metabolic network



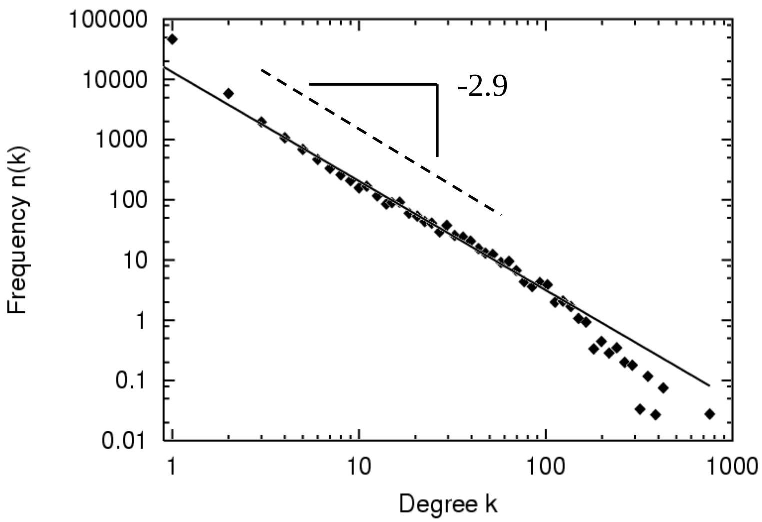

Organisms from all three domains of life are **scale-free** networks!

Jeong et al, Nature (2000)

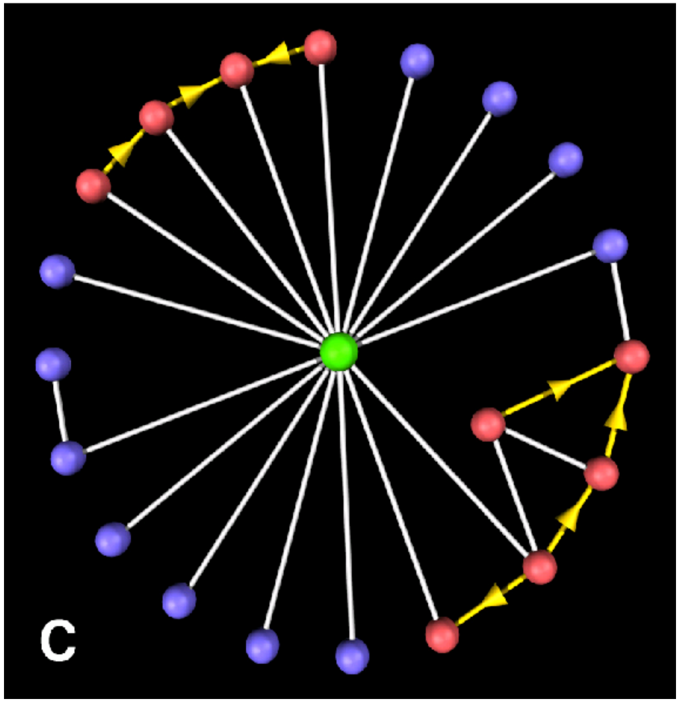
# More Examples

- Trust networks: Guardiola et al (2002)
- Email networks: Ebel etal PRE (2002)

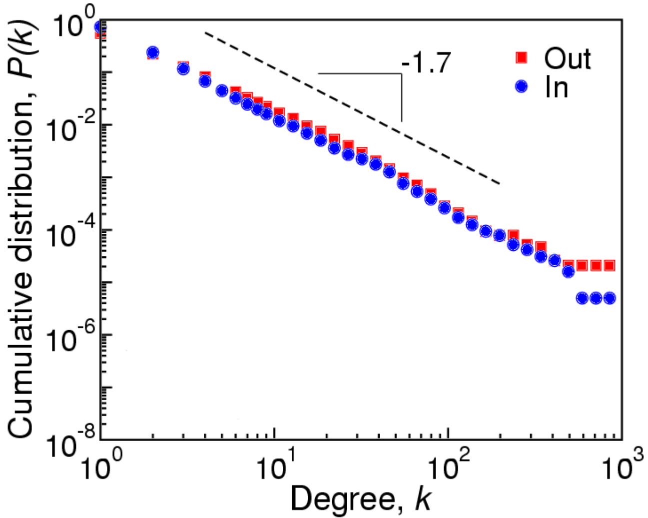
Email



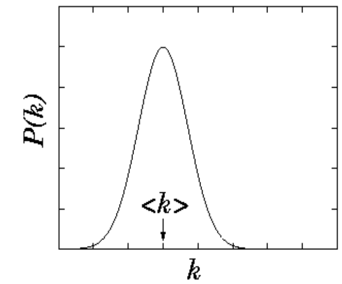
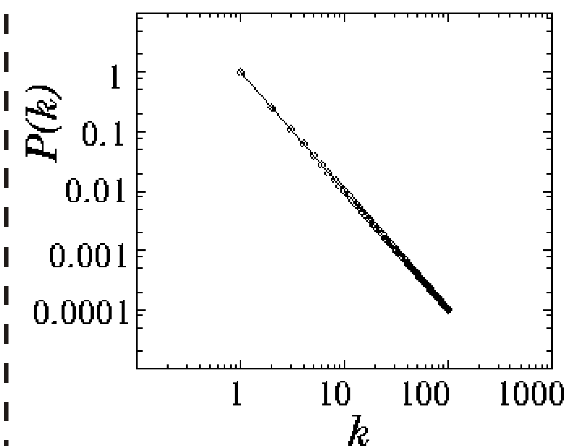

Trust




Trust



# Erdös Theory is Not Valid

| Stability and Immunization | Distance | Distribution | |
|---|---|---|---|
| $$q_c = 1 - p_c = 1 - \frac{1}{\langle k \rangle}$$ Critical concentration 30-50% | $$\boldsymbol{d} \sim \log \boldsymbol{N}$$ |  | Theory |
| (see table below) | Almost constant (Metabolic Networks, Jeong et. al. (Nature, 2000)) |  | Experiment |





| **Infectious disease** | Critical concentration |
|---|---|
| Malaria | 99% |
| Measles | 90-95% |
| Whooping cough | 90-95% |
| Fifths disease | 90-95% |
| Chicken pox | 85-90% |
| Mumps | 85-90% |
| Rubella | 82-87% |
| Poliomyelitis | 82-87% |
| Diphtheria | 82-87% |
| Scarlet fever | 82-87% |
| Smallpox | 70-80% |
| INTERNET | 99% |

**Generalization** of Erdös Theory:
Cohen, Erez, ben-Avraham, Havlin, PRL **85**, 4626 (2000)
**Epidemiology Theory**: Vespignani, Pastor-Satoral, PRL (2001), PRE (2001)

Cohen, Havlin,
Phys. Rev. Lett. 90, 58701(2003)

**Modelling**: Albert, Jeong, Barabasi (Nature 2000)

# Experimental Data: Virus survival

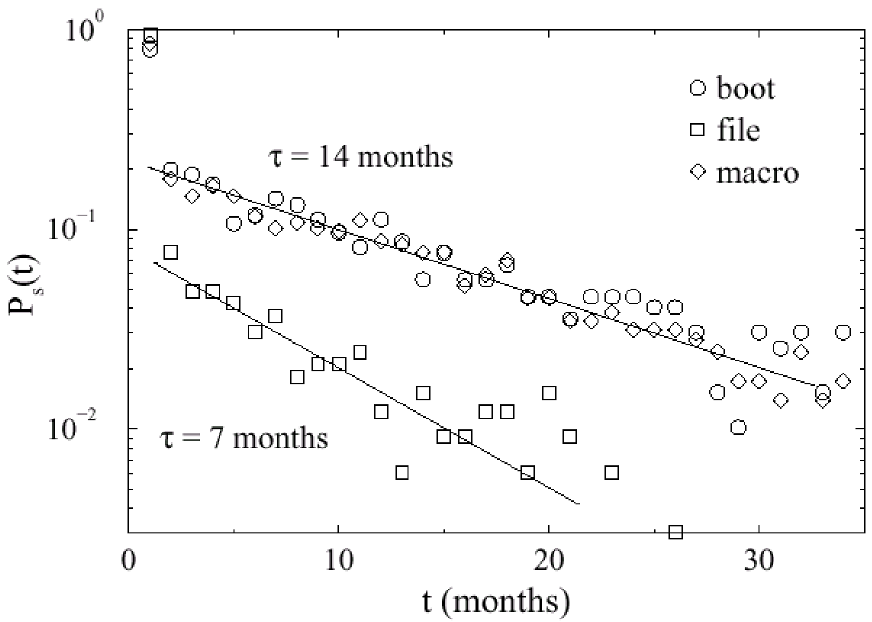
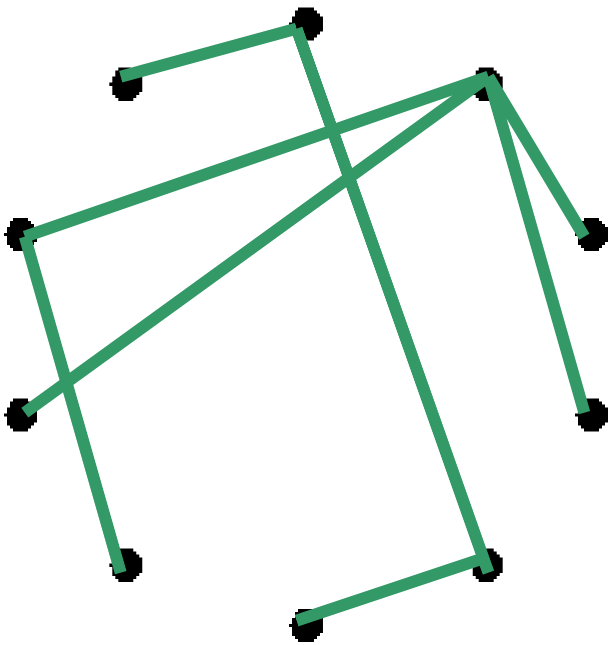



FIG. 1. Surviving probability for viruses in the wild. The 814 different viruses analyzed have been grouped in three main strains [9]: file viruses infect a computer when running an infected application; boot viruses also spread via infected applications, but copy themselves into the boot sector of the hard-drive and are thus immune to a computer reboot; macro viruses infect data files and are thus platform-independent. It is evident in the plot the presence of an exponential decay, with characteristic time $\tau \simeq 14$ months for macro and boot viruses and $\tau \simeq 7$ months for file viruses.

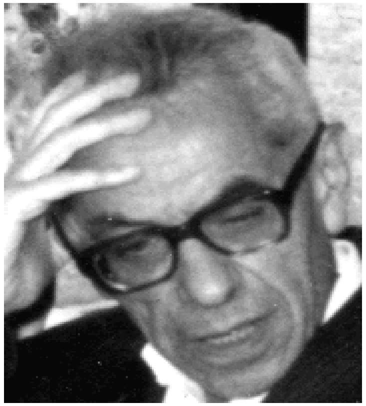
**(Pastor-Satorras and Vespignani, Phys. Rev Lett. 86, 3200 (2001))**

# Erdös-Rényi model
(1960)

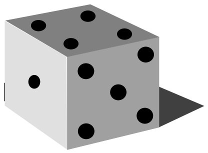


**Connect with probability p**

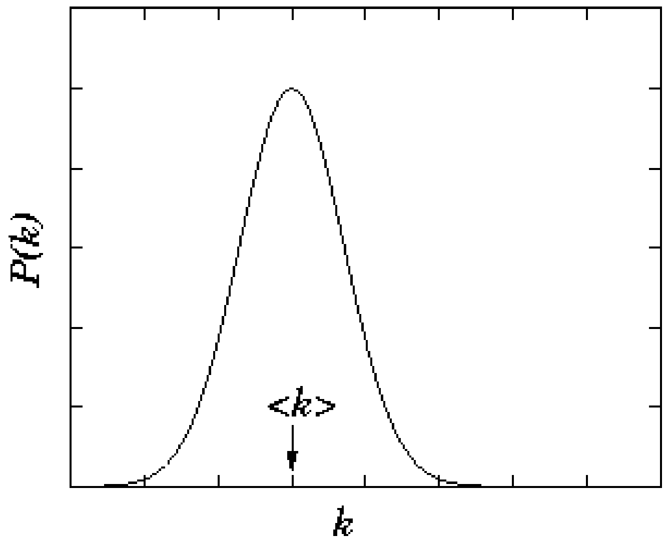
p=1/6
N=10
⟨k⟩ ~ 1.5



**Pál Erdös**
**(1913-1996)**



Poisson distribution




**- Democratic**

**- Random**

# Scale-free model

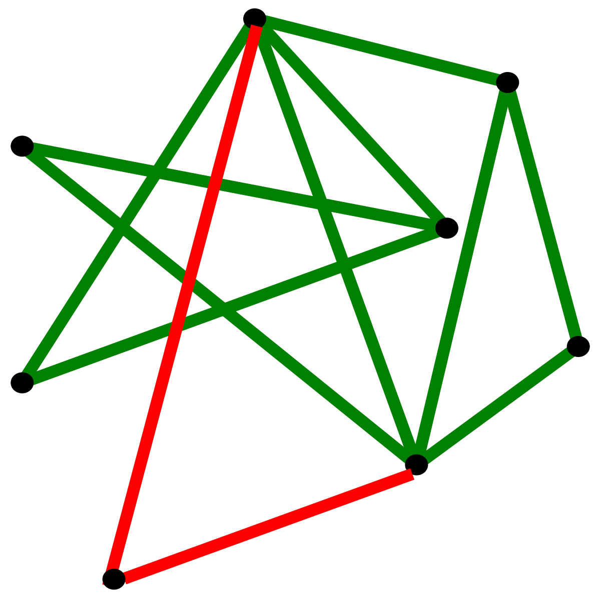
**(1) GROWTH :**

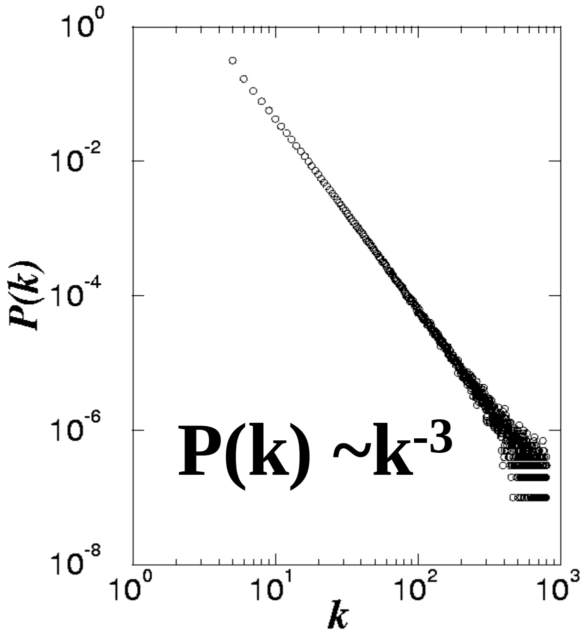
At every time step we add a new node with *m* edges (connected to the nodes already present in the system).

**(2) PREFERENTIAL ATTACHMENT :**

The probability $\Pi$ that a new node will be connected to node *i* depends on the connectivity $k_i$ of that node

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$






A.-L.Barabási, R. Albert, Science **286,** 509 (1999)

# Shortest Paths in Scale Free Networks

$$P(k) \sim k^{-\lambda}$$

$$d = const. \qquad \lambda = 2$$

**Ultra Small World**

$$\boxed{d = \log\log N \qquad 2 < \lambda < 3}$$

$$d = \frac{\log N}{\log\log N} \qquad \lambda = 3$$

(Bollobas, Riordan, 2002)

**Small World**

$$d = \log N \qquad \lambda > 3$$

(Bollobas, 1985)
(Newman, 2001)

Cohen, Havlin   Phys. Rev. Lett. 90, 58701(2003)

Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks

eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) chap.4

Confirmed also by: Dorogovtsev et al (2002), Chung and Lu (2002)

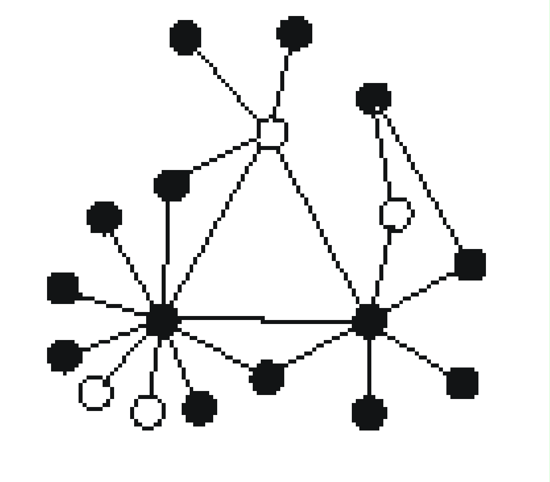
# Model of Stability

## Random Breakdown (Immunization)

The Internet is believed to be almost randomly connected scale-free network, where

$$P(k) \propto k^{-\lambda}, \lambda \approx 2.5$$

Nodes are randomly removed (or immune) with probability q=1-p

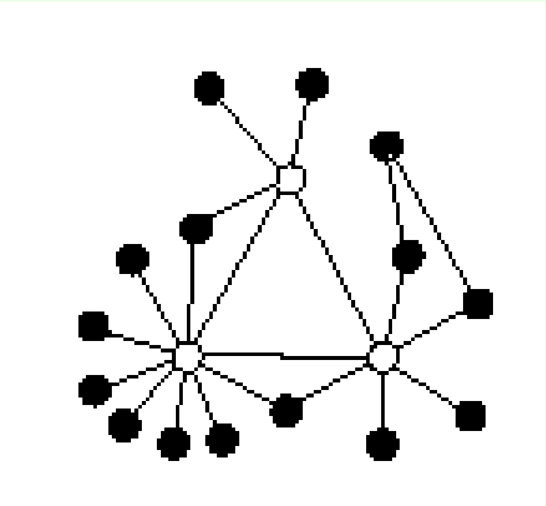


Where does the phase transition occur ?

# Model for Stability

## Targeted Attack (Immunization)

**The fraction, q , of nodes with the highest degree are removed (or immunized).**



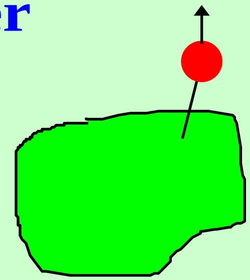
**Is this fundamentally different from random breakdown?**

We find that not only critical thresholds but also critical exponents are different !

THE UNIVERSALITY CLASS DEPENDS ON THE WAY CRITICALITY REACHED

## THEORY FOR ANY DEGREE DISTRIBUTION

# Condition for the Existence of a Spanning Cluster



**If we start moving on the cluster from a single site, in order that the cluster does not die out, we need that each site reached will have, on average, at least 2 links (one "in" and one "out").**

**This means:** $\langle k_i | i \leftrightarrow j \rangle = \sum_{k_i} k_i P(k_i | i \leftrightarrow j) \geq 2$**, where** $i \leftrightarrow j$ **means that site** $i$ **is connected to site** $j$ **.**

**But, by Bayes rule:** $P(k_i | i \leftrightarrow j) = \dfrac{P(i \leftrightarrow j | k_i) P(k_i)}{P(i \leftrightarrow j)}$

**We know that** $P(i \leftrightarrow j | k_i) = \dfrac{k_i}{N-1}$ **and** $P(i \leftrightarrow j) = \dfrac{\langle k \rangle}{N-1}$

**Combining all this together:** $\kappa \equiv \dfrac{\langle k^2 \rangle}{\langle k \rangle} = 2$

**(for every distribution) at the critical point.**

**Exponential graph:**

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle} = 2$$

$$\Rightarrow \langle k \rangle = 1$$

**Cayley Tree:**

$$p_c = \frac{1}{z-1}$$

Cohen et al, PRL 85, 4626 (2000): PRL 86, 3862 (2001)

# Percolation for Random Breakdown

**If percolation is considered the connectivity distribution changes according to the law:** $\bar{P}(k) = \sum_{k'>k} P(k') \binom{k'}{k} p^{k'-k}(1-p)^k$

**Calculating the change in** $\kappa$ **gives the percolation threshold:**

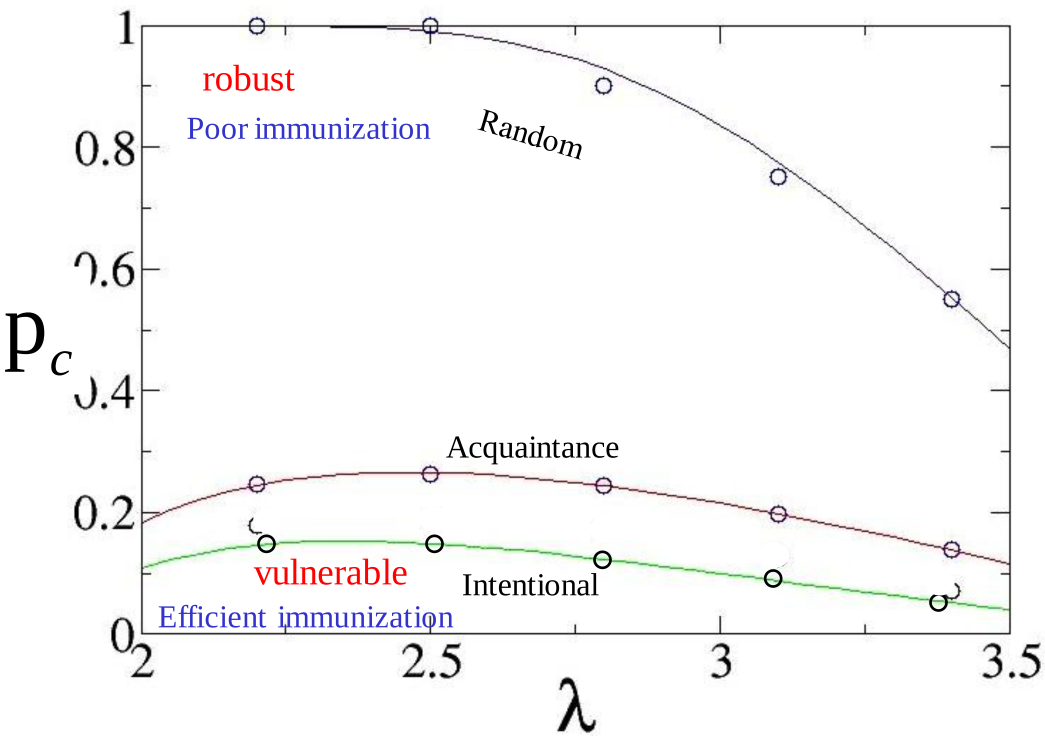
$p_c = 1 - q_c = \frac{1}{\kappa_0 - 1}$, **where** $\kappa_0 = \frac{\langle k_0^2 \rangle}{\langle k_0 \rangle}$. **compared to** $p_c = 1/ < k_0 >$ **for Erdos Renyi**

**For scale-free distribution with lower cutoff** $m$ **, and upper cutoff** $K$ **, gives**

$$\kappa_0 = \left(\frac{2-\lambda}{3-\lambda}\right)\frac{K^{3-\lambda} - m^{3-\lambda}}{K^{2-\lambda} - m^{2-\lambda}}, \quad K \sim N^{\frac{1}{\lambda-1}}.$$

For scale-free graphs where the second moment diverges.
No critical threshold!
Network is stable (or not immunized) even for q→1

# Critical Threshold
# Scale Free




Cohen et al. Phys. Rev. Lett. 91 , 168701 (2003)

General result:

$$p_c = \frac{1}{K_0 - 1}$$

$$K_0 \equiv \frac{\langle k^2 \rangle}{\langle k \rangle}$$

$$For \quad Poisson:$$

$$K_0 = \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle}$$

$$p_c = \frac{1}{\langle k \rangle}$$

Efficient Immunization Strategies:

Acquaintance Immunization

# Percolation for Targeted Attack (Immunization)

**Attack has two kinds of influence on the connectivity distribution:**

- **Change in the upper cutoff**

**Can be calculated by** $\sum_{k=\bar{K}}^{K} P(k) = p$ **,**

**or approximately:** $\bar{K} = mp^{1/(1-\lambda)}$ **.**

- **Change in the connectivity of all other sites due to possibility of a broken link (which is different than in random breakdown). The probability of a link to be removed can be calculated by:**

$$\bar{p} = \frac{1}{\langle k_0 \rangle} \sum_{k=\bar{K}}^{K} kP(k),$$

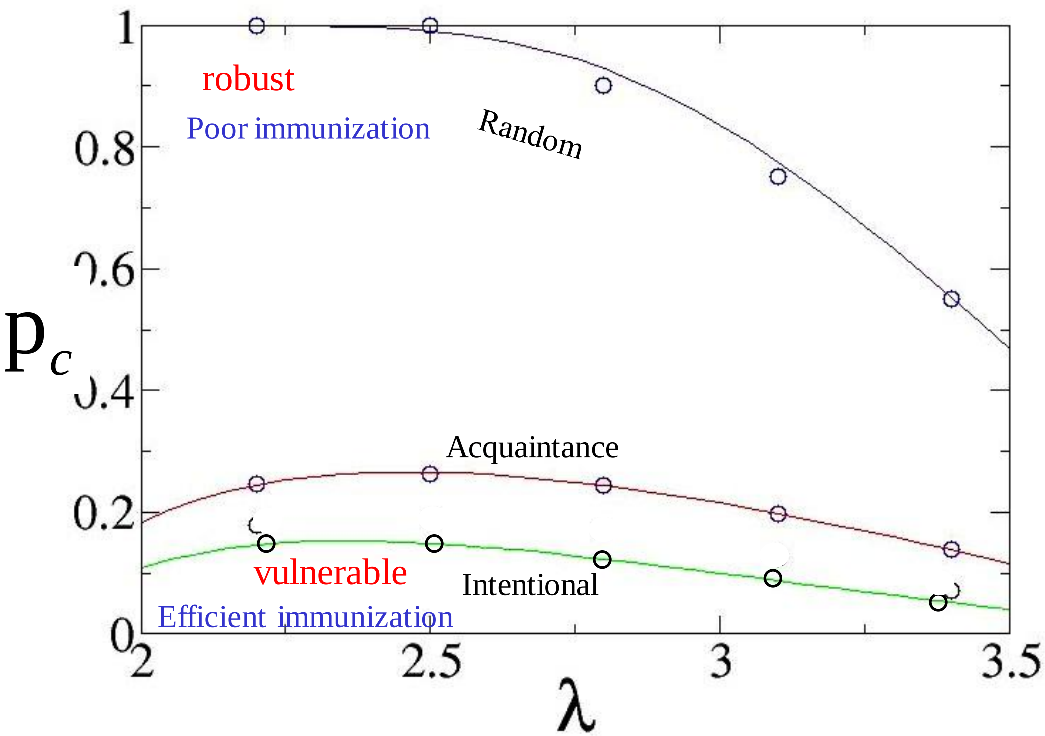
**or approximately:** $\bar{p} = p^{(2-\lambda)/(1-\lambda)}$ **.**

**Substituting this into:** $1 - \bar{p}_c = \frac{1}{\bar{k} - 1}$ **,**

**where** $\bar{\kappa} = \left(\frac{2-\alpha}{3-\alpha}\right) \frac{\bar{K}^{3-\lambda} - m^{3-\lambda}}{\bar{K}^{2-\lambda} - m^{2-\lambda}}$ **, gives the critical threshold.**

There exists a finite percolation threshold even for networks robust to random removal

# Critical Threshold
# Scale Free




Cohen et al. Phys. Rev. Lett. 91 , 168701 (2003)

General result:

$$p_c = \frac{1}{K_0 - 1}$$

$$K_0 \equiv \frac{\langle k^2 \rangle}{\langle k \rangle}$$

$$For \quad Poisson:$$

$$K_0 = \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle}$$

$$p_c = \frac{1}{\langle k \rangle}$$

Efficient Immunization Strategies:

Acquaintance Immunization

# Critical Exponents

**Using the properties of power series (generating functions) near a singular point (Abelian methods), the behavior near the critical point can be studied.**

**(Diff. Eq. Melloy & Reed (1998) Gen. Func. Newman Callaway PRL(2000), PRE(2001))**

**For random breakdown the behavior near criticality in scale-free networks is different than for random graphs or from mean field percolation. For intentional attack-same as mean-field.**

**Even for networks with $3 < \lambda < 4$, where $\langle k \rangle$ and $\langle k^2 \rangle$ are finite, the critical exponents change from the known mean-field result $\beta = 1$. The order of the phase transition and the exponents are determined by $\langle k^3 \rangle$ .**

**Size of the infinite cluster:**

$$P_\infty \sim (p - p_c)^\beta \qquad \beta = \begin{cases} \dfrac{1}{3-\lambda} & 2 < \lambda < 3 \\ \dfrac{1}{\lambda - 3} & 3 < \lambda < 4 \\ 1 & \lambda > 4 \end{cases}$$

(known mean field)

**Distribution of finite clusters at criticality:**

$$n_s \sim s^{-\tau} \qquad \tau = \begin{cases} \dfrac{2\lambda - 3}{\lambda - 2} & \lambda < 4 \\ 2.5 & \lambda \geq 4 \end{cases}$$

(known mean field)

# Fractal Dimensions

**From the behavior of the critical exponents the fractal dimension of scale-free graphs can be deduced.**

**Far from the critical point - the dimension is infinite - the mass grows exponentially with the distance.**

**At criticality - the dimension is finite for $\lambda > 3$ .**

**Chemical dimension:**

$$d_l = \begin{cases} \dfrac{\lambda-2}{\lambda-3} & \lambda < 4 \\ 2 & \lambda \geq 4 \end{cases}$$

$$S \sim \ell^{d_\ell}$$

**Fractal dimension:**

$$d_f = \begin{cases} 2\dfrac{\lambda-2}{\lambda-3} & \lambda < 4 \\ 4 & \lambda \geq 4 \end{cases}$$

$$S \sim R^{d_f}$$

**Embedding dimension:**

(upper critical dimension)

$$d_c = \begin{cases} 2\dfrac{\lambda-1}{\lambda-3} & \lambda < 4 \\ 6 & \lambda \geq 4 \end{cases}$$

Random Graphs – Erdos Renyi(1960)

Largest cluster at criticality

$$S \sim N^{\frac{2}{3}}$$

Scale Free networks

$$S \sim R^{d_f} \sim N^{\frac{d_f}{d_c}} \sim N^{\frac{\lambda-2}{\lambda-1}} \qquad \lambda \leq 4$$

$$S \sim N^{\frac{2}{3}} \qquad \lambda \geq 4$$

**The dimensionality of the graphs depends on the distribution!**