Scaling Theory

The distribution of clusters of size s per lattice site for Cayley tree:

$$n_s(p) = s^{-\tau} e^{-|p-p_c|^{1/\sigma} s}$$

with $\tau = 5/2, \ \sigma = 1/2$

- We assume that $n_s(p)$ retain the same scaling form for regular lattices

$$n_s(p) = s^{-\tau} f_{\pm}(|p-p_c|^{1/\sigma} s)$$

- The $\pm$ refer to below and above p_c.
- The critical exponents are universal and depend on dimension
- The form of $f(x)$ need not to be universal

Scaling Relations

Accepting the scaling ansatz the mean cluster size S and the probability P_∞ can be calculated.

$$S = \sum_{s=1}^{\infty} s\left(\frac{sn_s}{\sum_{s=1}^{\infty} sn_s}\right) = \frac{1}{p}\sum_{s=1}^{\infty} s^2 n_s$$

$$c \equiv |p - p_c|^{1/\sigma}$$

$$cs = z$$

$$\sim \int_1^\infty s^{2-\tau} f(cs)ds \sim c^{\tau-3}\int_c^\infty z^{2-\tau} f(z)dz$$

Assuming $\tau<3$ which is confirmed later, yield

$$S \sim |p - p_c|^{(\tau-3)/\sigma}$$

$$\gamma = \frac{3-\tau}{\sigma}$$

Scaling Relations

Calculating P_∞ :

Each site on the lattice is either:
(a) empty with prob. *1-p*
(b) occupied and on the infinite cluster with prob. pP_∞
(c) occupied but not on the infinite cluster with prob.

$$p(1-P_\infty) \equiv \sum_s sn_s$$

Below p_c, $P_\infty = 0$ and $p = \sum_s sn_s$

Thus,

$$P_\infty = 1 - \frac{1}{p}\sum_s sn_s$$

$$P_\infty = \frac{1}{p}\sum_s s(n_s(p_c) - n_s(p)) + (p - p_c)/p$$

The sum is proportional to: $P_\infty = c^{\tau-2}\int_c^\infty z^{1-\tau}(f(0) - f(z))dz$

$$c \equiv |p - p_c|^{1/\sigma}$$

$$cs = z$$

Thus $\beta = (\tau - 2)/\sigma$

Scaling Relations

P_∞ and S represent the first and second moment of n_s

The zeroth moment $M_0 \equiv \sum_s n_s$ represents the mean number of clusters (per lattice site).

$$M_0 \equiv \sum_s n_s \sim |p - p_c|^{2-\alpha}$$

This relation defines α (analogous to specific heat)

$$M_0 \equiv \sum_s n_s \sim \int_1^\infty s^{-\tau} f(|p-p_c|^{1/\sigma} s)ds \sim c^{\tau-1} \int_c^\infty z^{-\tau} f(z)dz$$

$$2 - \alpha = (\tau - 1)/\sigma$$

For Cayley tree $\alpha = -1$

For $\alpha, \beta, \gamma, \nu$ and σ we have three relations.

Thus only two exponents are independent