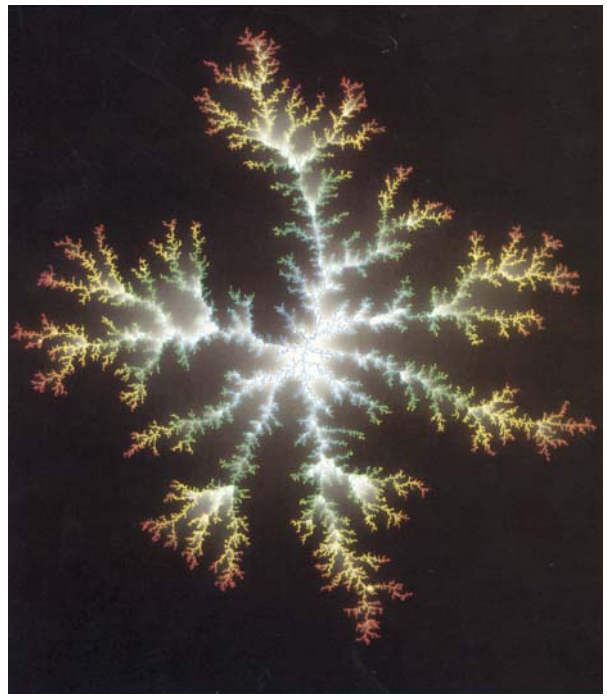
Percolation Chemical Dimension

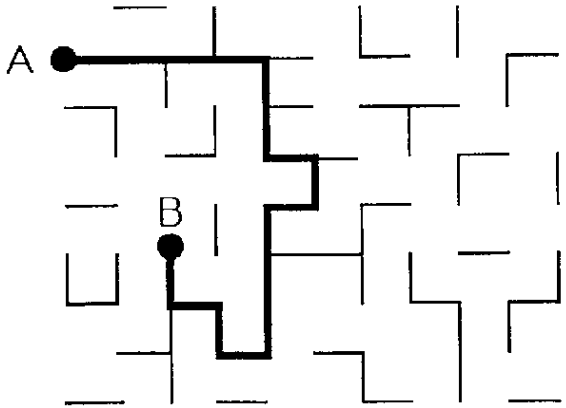
- The fractal dimension d_f is not enough to characterize the percolation cluster

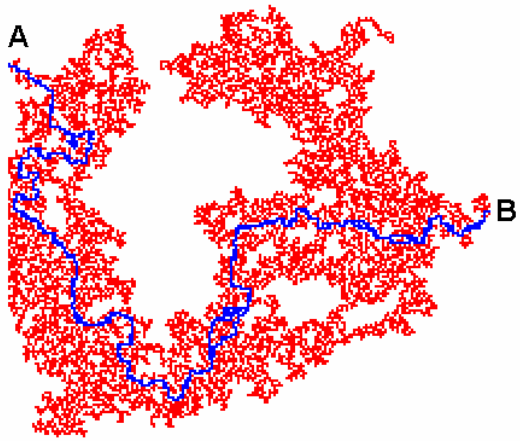


- This is obvious when looking on DLA and percolation
- For d=3 both have same $d_f \simeq 2.5$
- Percolation has loops – DLA has no loops

Shortest Path

- For better characterization we study the shortest path between A and B





- The shortest path ℓ is self-similar with fractal dimension d_{min}

$$\ell \sim r^{d_{min}}$$

Chemical Dimension

- The chemical dimension d_ℓ is defined by

$$M \sim \ell^{d_\ell} \sim r^{d_{\min} \cdot d_\ell}$$

Since $M \sim r^{d_f} \Rightarrow \boxed{d_f = d_{\min} \cdot d_\ell}$

- For percolation in $d = 2$ $d_{\min} \simeq 1.13$

$d = 3$ $d_{\min} \simeq 1.38$

Only numerical simulations ! No theory !

- For $d \geq 6$ $d_{\min} = 2$

Theory: in ∞ dimensions - no interactions
each path is a random walk

$$\ell \sim r^2 \qquad (\ell \text{ - is like time})$$

- $d_{\min}$ distinguish between DLA and percolation

$d_{\min} = 1.38$ for percolation (d=3)

$d_{\min} = 1$ for DLA (d=3)

Disease and fire spreading

- The chemical distance is important for describing disease and fire spreading
- Assume sick people or trees are on a lattice with concentration p
- At each stage one chemical shell is infected
- The total number of trees burned until time $t = \ell$ is

$$M(t) = t^{d_\ell}$$

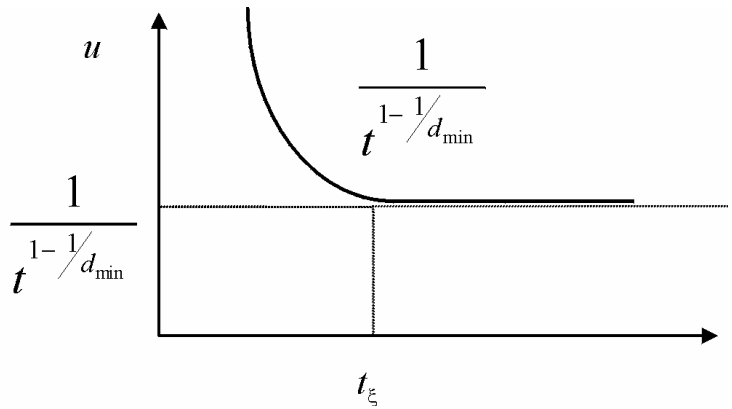
	2		
X	1	X	
1	0	1	2
2	X	X	X

- The distance where the disease or fire reached is

$$r \sim t^{1/d_{\min}}$$

- The velocity of spreading is

$$u = \frac{dr}{dt} \propto t^{\frac{1}{d_{\min}}-1}$$



- This is true only for $r \sim t^{1/d_{\min}} < \xi$

For $r > \xi$ $r \sim t$ (regular lattice)

$\Rightarrow$ $u =$ **constant**

- The constant velocity will occur at

$$r = \xi = (p - p_c)^{-\nu} \text{ and } t_\xi \sim \xi^{d_{\min}} \sim (p - p_c)^{-\nu d_{\min}}$$

➢We can calculate the constant velocity:

From $u \simeq t^{\frac{1}{d_{\min}}-1}$

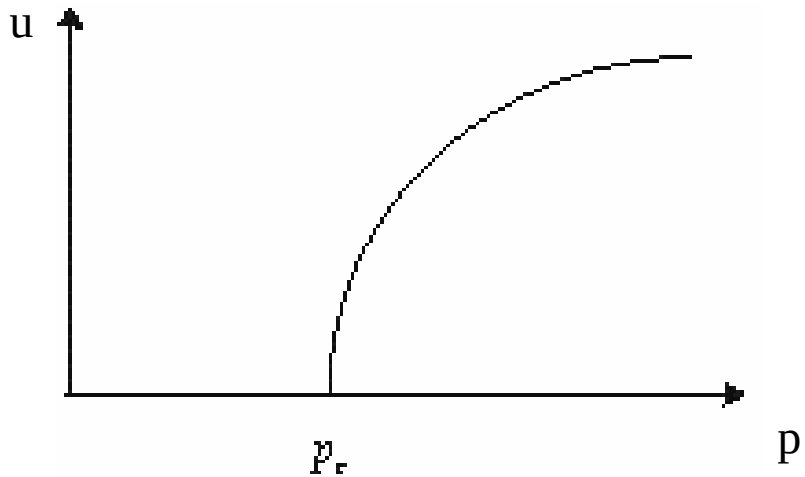
$$u \simeq t_{\xi}^{\frac{1}{d_{\min}}-1} \simeq (p-p_c)^{-\nu d_{\min}\left(\frac{1}{d_{\min}}-1\right)}$$

$$\boxed{u \simeq (p-p_c)^{(d_{\min}-1)\nu}}$$

➢For $d=2$ $\quad (d_{\min}-1)\nu = 0.13 \cdot \frac{4}{3} \cong 0.16$

- very small exponent

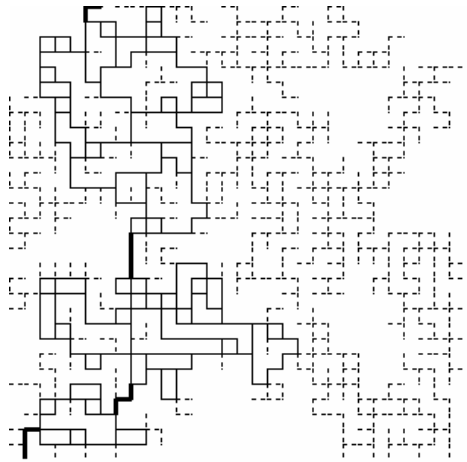
➢If at $p<p_c$ the fire (or disease) does not spread at all
just above p_c the fire (or disease) spreads very fast which increases when p increases



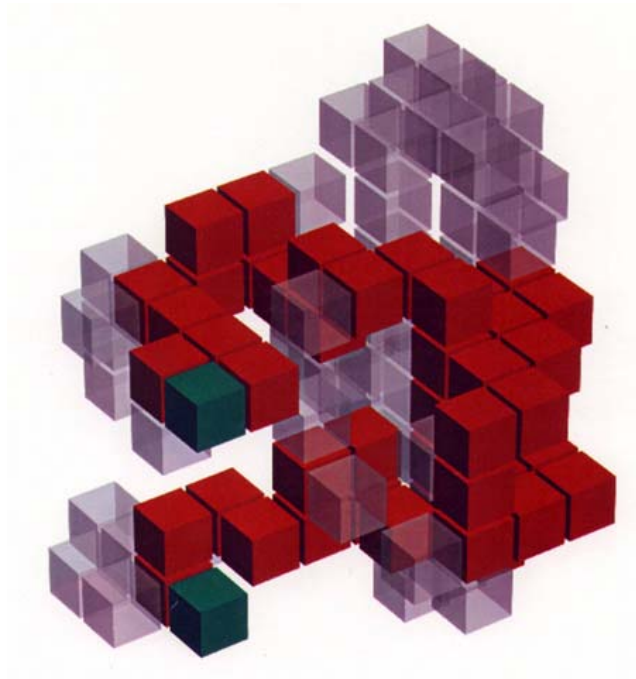
Percolation – substructures

Backbone, dead ends, red bond and blobs

- The fractal dimensions d_f and $d_{\min}$ are not enough to characterize percolation clusters
- We impose a voltage drop between two sites on the infinite cluster
- The backbone includes all bonds which carry current
- The dead ends are the parts that do not carry any current
- The red bonds (called also singly connected bonds) are those links that carry all current (dark black). When they are cut the current stops
- Blobs are the parts of the backbone left after removing the red bonds



Backbone



- The backbone for site percolation in d=3 is shown here

- Green sites have a voltage drop
  The backbone is shown in red
  The dead ends are shown in gray

- The backbone in any dimension is only a small (zero) fraction of the infinite cluster. Its fractal dimension is smaller:

$$M_{BB} \sim R^{d_B}, \; d_B < d_f \qquad \text{(see Table)}$$

- Thus, most of the mass of the cluster is in the dead ends.

Backbone

- The values of the fractal dimension of the backbone, d_B are known only numerically (for $2 \le d \le 5$). Analytical derivation exists only for $d = d_c = 6$ (see Table)

- $d_{\min}$ on the backbone is the same as $d_{\min}$ on the percolation cluster. This is since from every two sites one can generate a backbone and the shortest path will be on it.

- d_l^B - the chemical dimension of backbone is not the same as d_l for the cluster.

$d_l^B = d_B / d_{\min}$ (while $d_l = d_f / d_{\min}$)

d	2	3	6
d_f	91/48	2.53	4
$d_{\min}$	1.1307	1.374	2
d_{red}	3/4	1.143	2
d_h	7/4	2.548	4
d_B	1.6432	1.87	2

Red bonds

- The fractal dimension of red bonds is known analytically (Coniglio 1982)

- For any d: the number of red bonds is

$$n_{red} \sim (p - p_c)^{-1}$$

- Since the correlation length $\xi \sim (p - p_c)^{-\nu}$ it follows

$$n_{red} \sim \xi^{1/\nu}$$

- When looking at $r < \xi : n_{red} \sim r^{1/\nu}$

- Thus, the fractal dimension of red bonds is for all d

$$d_{red} = \frac{1}{\nu}$$