Distance distribution in extreme modular networks

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Modularity is a key organizing principle in real-world large-scale complex networks. Many real-world networks exhibit modular structures such as transportation infrastructures, communication networks, and social media. Having the knowledge of the shortest paths length distribution between random pairs of nodes in such networks is important for understanding many processes, including diffusion or flow. Here, we provide analytical methods which are in good agreement with simulations on large scale networks with an extreme modular structure. By extreme modular, we mean that two modules or communities may be connected by maximum one link. As a result of the modular structure of the network, we obtain a distribution showing many peaks that represent the number of modules a typical shortest path is passing through. We present theory and results for the case where interlinks are weighted, as well as cases in which the interlinks are spread randomly across nodes in the community or limited to a specific set of nodes.

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I. INTRODUCTION

The study of complex networks gains extensive interest in the last years as networks successfully model and lead to better understanding of many real world systems and processes in which interacting objects are involved. In these models, objects are represented as nodes, and the interactions by links [1–7].

Many real world networks exhibit a modular or community structure [8–11]. That is, a network is comprised of smaller networks (called communities, or modules) that are highly connected within themselves (by intralinks), and have a lower number of links between them (interlinks), which is a key to their structure and function. For demonstration, see Fig. 1. Knowing the distances distribution within networks with such topology is important for many reasons such as designing fast-communication, navigation, disease spreading, and for optimizing processes on large graphs.

For each random pair of nodes i and j in the network, many paths can exist, or none at all. The distance between a pair of nodes is naturally defined as the shortest path length among all the paths existing between them. Distribution of shortest paths are expected to depend on the network structure and size. However, apart from a few studies [12–17], the shortest paths length distribution (DSPL) despite its importance, attracted little attention. Recent studies developed analytical methods to compute the DSPL in Erdős-Rényi and configuration-model networks [18,19]. Another paper studied the DSPL in modular random networks [20], testing the conditions in which the number of interlinks between two or more modules control the network topology. This means, answering the question "how many links between two modules are needed in order to unite them into one?" Adding more interlinks results in a change of the SPL distribution, which approaches a δ function as we add more interinks. Still, the case where the connections between the modules is itself a complex network, meaning that the interlinks are determined according to a given outer network, an analytical approach for finding the DSPL has not been developed yet.

As a motivation for the present study, we analyzed the distance distribution (DSPL) in the internet routers-IP autonomous systems (AS) network. Each AS functions as a community, and contains routers IPs which are the nodes inside the community. The data were obtained from the center of applied Internet analysis (Caida) [21]. Several studies have been performed on distances in the internet [22–24]. Here, we want to point out a specific phenomenon which occurs when more and more interlinks are removed. In this case, a wavy distribution emerges. In Fig. 2 we show the distance distribution (DSPL) of our data for different values of maximal interlinks degrees. By limiting the number of interlinks of an AS, we mimic a situation in which the internet network undergoes an attack or power shortage. We can observe in Fig. 2 multiple peaks for the DSPL after such an attack, representing the modules passed by the shortest path. This phenomenon motivates us here to develop a simplified model of extreme community structure which exhibits a wavy DSPL, and we study this analytically in order to better understand this phenomenon.

In this paper we develop an analytical approach for obtaining the DSPL of a modular network. Our theory calculates

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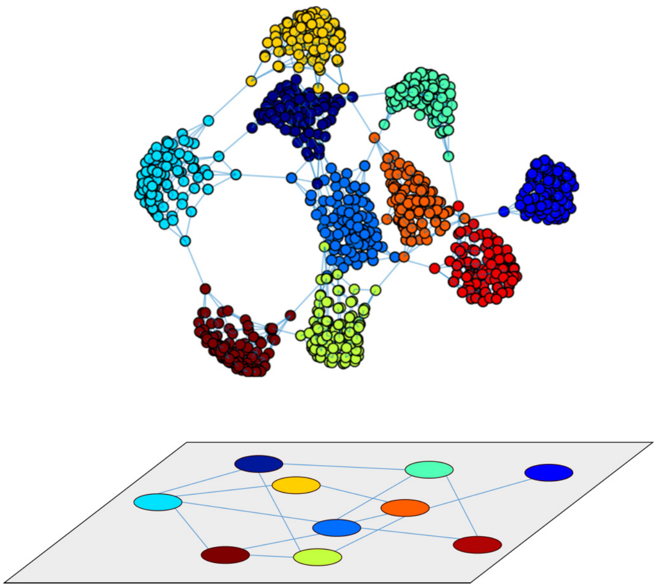

FIG. 1. Illustration of the model. We study networks that comprise M modules, each of which has the same degree distribution, same topology, and same size, n. Thus, we study a modular network with a total of $M \times n = N$ nodes. The upper picture shows the whole network, where nodes of each module have different color. The lower picture is a projection of the upper network, showing the outer network, where each node represents a module. We start by constructing each module according to a given topology, mean degree ($k_{\rm in}$) and links distribution, and then construct the outer network which has its own topology, mean degree ($k_{\rm out}$), and links distribution, where we consider each module as a node. Two modules can be connected by a maximum of one link. In Sec. III we provide an analysis of different network topologies.

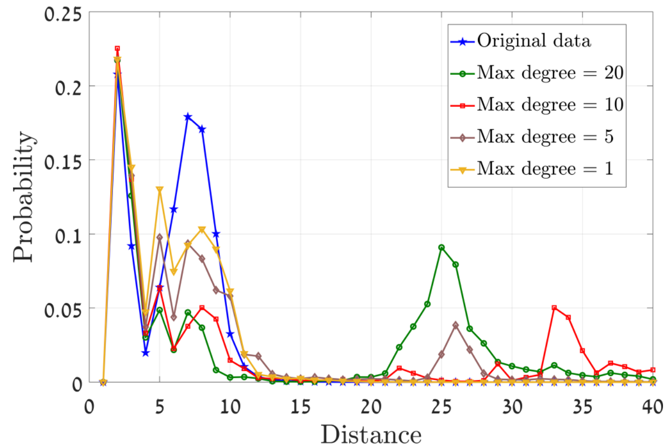

FIG. 2. Analysis of autonomous systems (AS) distances distribution for different values of maximal number of interlinks outgoing from one module (max degree). AS data were obtained from Caida and collected using MIDAR-iff, where router topology based on aliases discovered by MIDAR, iffinder, and kapar. The data contained approximately 100 million nodes which are assigned into 47 000 communities (AS). a node that connects two different ASs is an interlinked node. Here, we consider a case in which the internet network undergoes a deliberate attack or experience node failures due to lack of electricity supply on interlinks between AS. We expect that interlinked nodes fall with higher probability, due to their high between-ness centrality (in case of a deliberate attack) or high power demand in the case of power outage). We examine the distance distribution for those cases, allowing different maximal interlink degree of an AS. Here, distance is defined as the shortest path length between two nodes in the graph. Note the wavy pattern of the distribution.

the DSPL of the network given the shortest path distribution within one module (in net), and the DSPL of the network that connects the modules by treating each module as a node (out net). Our method holds for any inner and outer network topologies, and not only for random networks. The model we suggest assumes an extreme community condition where each module I is connected to module J with a maximum of one link that connects two randomly chosen nodes in both modules. Another condition we assume here is that the outer network has no small loops as explained in detail below. In order to better simulate real world phenomena, such as routing and transportation between cities or countries, our model assumes a weight w for interlinks, where intralinks weight is set to 1. We further include analysis of various cases in which interlinks are limited to a specific set of nodes rather than being chosen randomly from the inner network. Analytic analysis of specific network topologies is also included.

The paper is organized as follows. In Sec. II A we present the basic model and theory we use to find the DSPL. In Secs. II B and II C we extend the theory for different interlinks configurations. In Sec. III we present our results of DSPL, from both our theory and simulations of selected network topologies. More comprehensive mathematical analysis of some specific cases of network of networks is presented in more detail in the Appendix.

II. MODEL AND THEORY

A. Basic model

Let a network consist of m communities, or modules. Each module is assumed to be of the same size and constructed in the same fashion (or just with the same distances distribution), e.g., Erdős-Rényi, scale-free (SF), random regular (RR), lattice, or any other structure. An outer network, which also can take any structure, regards every module as a node. Therefore we obtain a "large" network which comprises modules, and another network on top of it which connects those modules as illustrated in Fig. 1.

Our model assumes the following:

(1) There is at most one interlink between two modules.

(2) The interlinks connect between pairs of *random* nodes of two modules.

(3) Interlinks have a weight w (integer), while the weight of intralinks is 1.

(4) The outer network has no small loops.

(5) As a consequence of (4), an outer shortest path between modules in the outer network is single and the second outer shortest path is much longer than it. Therefore, the shortest path in the whole network, in most cases, will pass through the shortest path of the outer network.

It is important to notice that while assumptions (4) and (5) hold for short distances, they partially fail for the long distances in the network. Hence, we expect slight deviations

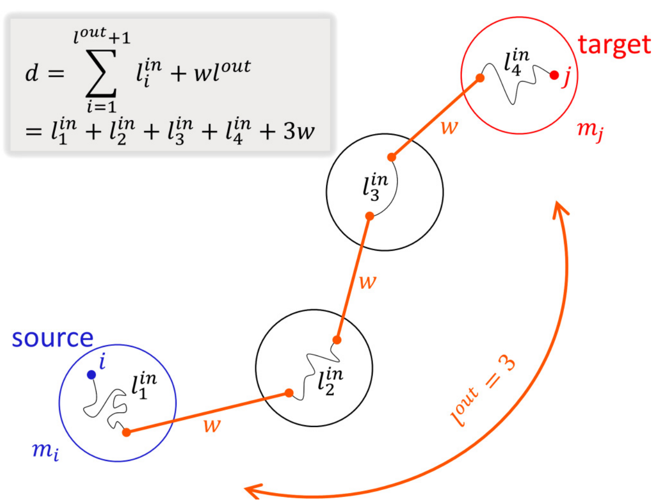

FIG. 3. Illustration of the problem and the theory. Consider two random nodes i and j, which reside in different modules m_i and m_j. In order to reach via the shortest path from node i (source) to node j (target), one has to walk as follows. First, to find the outer shortest path that connects the modules (l^{out}). Next, to look for the shortest path within m_i to the node that connects the source node to a node that resides in the next module of l^{out}, which is denoted by l_1^{in} in the figure. We iterate this process again in the next modules on the path, until we finally land in our target node. Our total path length will be $d = \sum l_i^{\text{in}} + wl^{\text{out}}$.

at the end of the distribution, as seen in general in the figures. Random sparse networks, for instance, exhibit locally tree-like behavior [25]. The range of this behavior is up to the average distance of the network approximately [26], therefore for these networks our theory is accurate up to the average distance of the network, and then it has slight deviations as we show below. For a one-dimensional (1D) lattice, for example, assumptions (4) and (5) are valid up to the longest distances, whereas for a two-dimensional (2D) lattice, the assumptions fail.

Now, the shortest path length (SPL) distribution in each module (inner paths) is P_l^{in}, and has the generating function $G_{\text{in}}(x) = \sum_{l=0}^{\infty} P_l^{\text{in}} x^l$. Likewise, the SPL distribution of the outer network is P_l^{out} and has the generating function $G_{\text{out}}(x) = \sum_{l=0}^{\infty} P_l^{\text{out}} x^l$.

According to the above assumptions, one can find that the SPL between two random nodes in the network satisfies

$$d = \sum_{i=1}^{l^{\text{out}}+1} l_i^{\text{in}} + wl^{\text{out}}, \tag{1}$$

which yields

$$d + w = \sum_{i=1}^{l^{\text{out}}+1} \left(l_i^{\text{in}} + w\right), \tag{2}$$

where d is the total distance between two random nodes, l^{out} is the external shortest path length between the communities those nodes reside in, and l_i^{in} is the internal distance between nodes in the same community which function as the connecting nodes between the communities in l^{out}. See illustration in Fig. 3.

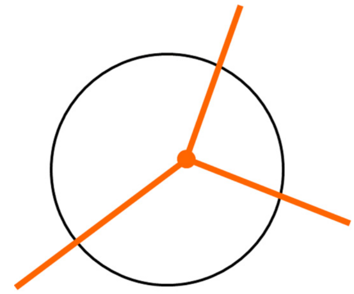

FIG. 4. Illustration of the model in Sec. II B. Here, each module has only one interconnected node to which all the interlinks of this module are connected.

This is a sum of independent random variables where the number of elements of the sum itself, is also a random variable. Then we can use known theorems [27] to conclude the following results.

First, from Wald's identity,

$$\langle d + w\rangle = \langle l^{\text{out}} + 1\rangle\langle l^{\text{in}} + w\rangle,$$

which gives

$$\langle d\rangle = (\langle l^{\text{out}}\rangle + 1)(\langle l^{\text{in}}\rangle + w) - w. \tag{3}$$

This result suggests that in small world networks the extreme modularity condition makes the average distance much longer. Furthermore, for the generating functions one can write

$$[x^w G_d(x)] = [xG_{\text{out}}(x)] \circ [x^w G_{\text{in}}(x)],$$
$$x^w G_d(x) = x^w G_{\text{in}}(x) G_{\text{out}}[x^w G_{\text{in}}(x)],$$

where $G_d(x)$ is the generating function of P_d, the probability distribution of d, and $\circ$ is a composition of functions.

Thus, we get

$$G_d(x) = G_{\text{in}}(x) G_{\text{out}}[x^w G_{\text{in}}(x)]. \tag{4}$$

Since we have the generating function of the shortest path distribution we are consequently able to find P_d by derivation or integration (Cauchy formula) numerically by

$$P_d = \frac{G_d^{(d)}(0)}{d!} = \frac{1}{2\pi i} \oint \frac{G_d(z)}{z^{d+1}} dz, \tag{5}$$

where the integral is performed on a close path around $z = 0$ in the complex plain. This integral is far more simple to compute numerically than computing high derivatives. A simple contour can be a canonical circle with $r = 1$.

See Appendix A where we analyze analytically few specific cases of network of networks topologies: including, 1D and 2D lattices, Poisson distance distribution, two modules, and star graph. We find for these cases explicitly all or part of the following expressions. $G_{\text{in/out}}(x)$, $G_d(x)$, and P_d. For two modules with Poisson DSPL we find analytically also a condition for the appearance of two peaks rather than one peak, see Eqs. (A19) and (A20) and Figs. 12 and 13.

B. One node has all the interlinks in each module

When analyzing the internet data (AS) that was mentioned above, we noticed the fact that many interconnected nodes have multiple interlinks. In order to cover other realistic cases

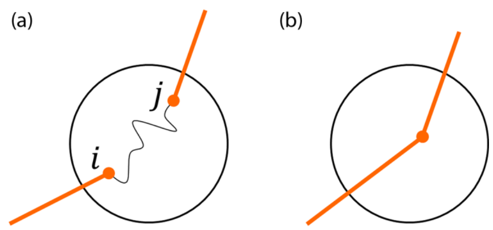

FIG. 5. Illustration of the model in Sec. II C. (a) In this scenario we enter the community through node i and leave the community with probability p through a different node j, with addition of a tour inside the community. (b) Here, with probability $1-p$, the arrival and the departure to and from the community is from the same node.

such as this, we consider also the scenario in which all the interlinks of a module go out and in from the same single interconnected node, rather than from random nodes as the above model, see Fig. 4. This situation changes the distance significantly,

$$d = l_1^{\rm in} + l_2^{\rm in} + w l^{\rm out},$$

and if $l^{\rm out} = 0$ then $d = l^{\rm in}$ (because the source and the target reside in the same module). Hence,

$$G_d(x) = G_{\rm out}(0)G_{\rm in}(x) + [G_{\rm out}(x^w) - G_{\rm out}(0)][G_{\rm in}(x)]^2. \quad (6)$$

C. Different cases of interlinks connections

In this section, we consider the case in which, when entering a module via an interconnected node i, we leave this module via different interconnected node j with probability p, or, when departing the module via the same node with probability $1-p$. See Fig. 5.

In Appendix B we find that for this case

$$G_d(x) = G_{\rm out}(0)G_{\rm in}(x) + [G_{\rm in}(x)]^2 \frac{G_{\rm out}(x^w(1-p+pG_{\rm in}(x))) - G_{\rm out}(0)}{1-p+pG_{\rm in}(x)}. \quad (7)$$

One can see that the last equation converges nicely to those of Secs. II A [Eq. (4)] and II B [Eq. (6)] at the limits $p = 1$ and $p = 0$, respectively.

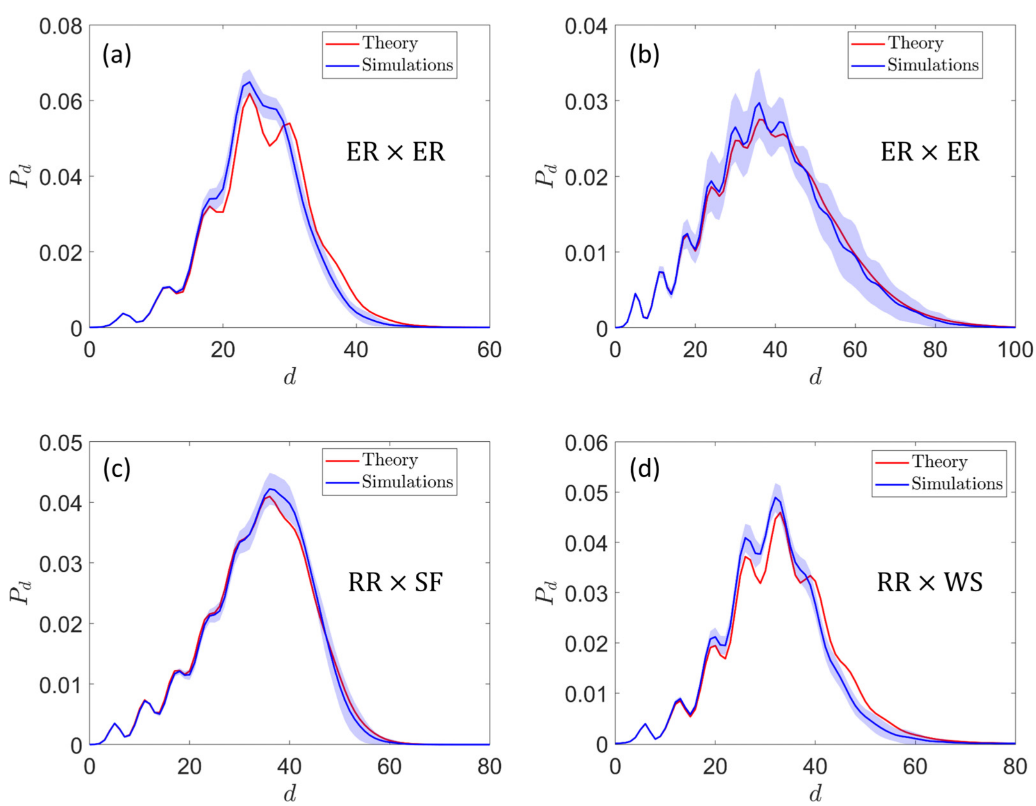

FIG. 6. Results of distance distribution in several types of modular networks with uniform links weight ($w = 1$). In (a) we show the case of ER×ER (outer network ER, modules ER), with $M = 10^2$ (number of modules), $k_{\rm out} = 4$, $n = 10^3$ (size of each module), and $k_{\rm in} = 4$. Theory, simulations mean and simulations standard deviation (shaded area) results are shown. (b) The same as (a) except that $k_{\rm out} = 2$. In (c) we show RR of SF (out: RR; in: SF), where $M = 10^2$, $k_{\rm out} = 3$, $n = 10^3$, $k_0^{\rm in} = 2$, and the power-law degree exponent is $\gamma_{\rm in} = 3$. In (d) we show RR of WS (Watts-Strogatz model), where $M = 10^2$, $k_{\rm out} = 3$, $n = 10^3$, $k_{\rm in} = 4$, and $\beta = 0.5$. Simulation results were taken over ten realizations. In all cases one can see a good agreement between theory and simulations except for slight deviations at large distances, the reason of which is discussed in the text. The wavy distributions found here are significantly wider than a distribution of a single network, due to the extreme modular structure. In a single ER network, the mean distance is given by $\langle d\rangle \approx \ln(N)/\ln(k)$, where in our model in the case of ER×ER the mean distance is $\langle d\rangle \approx (\ln(n)/\ln(k_{\rm in})+1)(\ln(M)/\ln(k_{\rm out})+1)-1$. See Eq. (3).

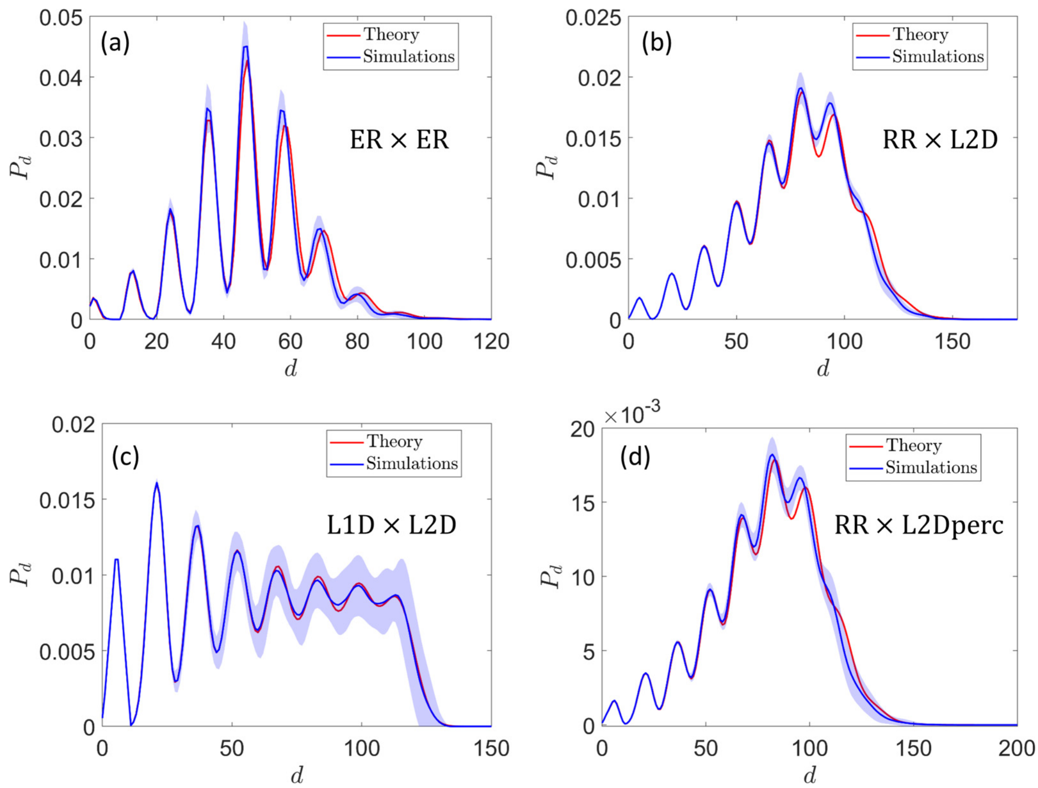

FIG. 7. Weighted interlinks cases. Assuming a weight w to interlinks, results for random and lattice networks are shown. In (a) we show a ER of ER network with the parameters $M = 10^2$, $k_{\text{out}} = 3$, $w = 10$, $n = 10^3$, and $k_{\text{in}} = 3$. Theory, simulations mean, and simulations standard deviation (shaded area) results are shown. In (b) we show a RR of L2D (outer: RR; inner: 2D lattice) network with the parameters $M = 10^2$, $k_{\text{out}} = 3$, $w = 10$, and $n = 10^2$. In (c) we show L1D of L2D (outer: 1D lattice; inner: 2D lattice) with $M = 15$, $w = 10$, and $n = 121$ and the theory is from the explicit formula in the Appendix. In (d) we show a RR of L2Dperc (2D lattice with percolation where fraction q of random nodes was removed) network with the parameters $M = 10^2$, $k_{\text{out}} = 3$, $w = 10$, $n = 10^2$, and $q = 0.2$. For this case, we test the DSPL in the giant connected component. All cases show exclusive distances distribution and good agreement between theory and simulations.

III. RESULTS

Figure 6 compares between theory and simulations for different network layouts and parameters where the interlinks have the same length as the inner links, i.e., $w = 1$. In general the figure shows a good agreement between theory and simulations.

It is important to notice the distance distribution exhibits a wavy behavior on top of a hill envelope. The intuitive explanation for this is that each hill represents paths between nodes in two modules that have the same outer distance. The first hill comes from paths between nodes inside the same module, while the second hill comes from paths between neighboring modules, which are about twice longer due to their consistency of two inner paths—the first, in the source module, from the source node to the interconnected node inside the source module, and the second, from the interconnected node in the target module, to the target node. The second hill is higher because there are more paths between neighboring modules than paths within a single module. In other words, in the outer network (in between modules), there are more shortest paths with $l^{\text{out}} = 1$ than with $l^{\text{out}} = 0$. The same holds for the third hill ($l^{\text{out}} = 2$) and so on. That is to say, what rules the hills' heights is the outer SPl distribution, therefore we get a bell shaped envelope which comes from the outer network distribution, and upon it hills which come from inner networks distribution.

Note that, for the long distances there is a slight deviation between the theory and the simulations results. This can be explained by the fact that in theory we neglect loops in the outer network, while in practice, for finite networks, there are long loops (the short ones are negligible). The long loops causes that there are modules far from each other have few outer similar paths between them. This multiplicity of similar outer paths shortens the distance from a source node to a target node because the shortest path is chosen among them. In this case, we will need to find the minimum of similar independent random variables, which is different (lower) than the expectation value relative to the random variables.

Figure 7 shows the results for the distance distribution for the extreme modular network, both theory and simulations, where interlinks are weighted with $w = 10$. It can be seen that the separation between the hills becomes more significant because paths between modules with different outer distances have dramatically different lengths as a result of the length of the interlinks. Within a single 2D lattice there is a broad distance distribution because the system is not a small world network. As a result when the interlinks are not much longer

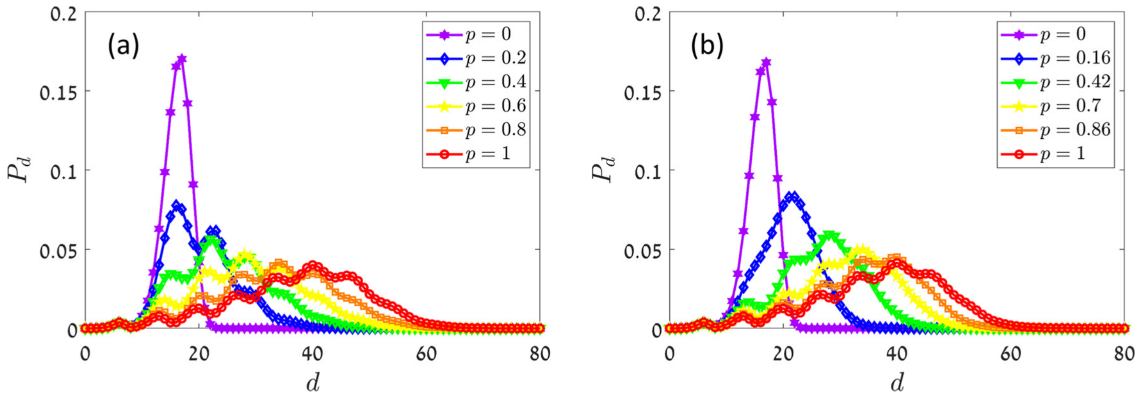

FIG. 8. Impact of the parameter p in both theory (a) and simulations (b). Results of Eq. (7) where the network is formed as follows. Out: RR, $M = 100$, $k = 3$, $w = 1$. In: RR, $n = 1000$, $k = 4$. where (a) shows the theory, Eq. (7), and (b) shows simulation results. The values of p in simulations were found by sampling many realizations of random shortest paths and measuring how many times each path goes in and out a module through different interconnected nodes and how many times via the same node. Results show good agreement between theory and simulations.

than the inner ones, the wavy behavior vanishes because the widths of hills are large so they become blended together. However, when w is sufficiently large, the waves are very distinct.

Figure 8 shows the results of Eq. (7), for various values of p. This model suits a more realistic case, in which there is a probability p of accessing and leaving a community through a different or the same $(1 - p)$ interconnected node. Note the reduction in the number of waves when p approaches 0, which is the case in which no intramodule paths were taken.

In order to examine the emergence of the wavy distribution, we regulate the parameter n, modules size. We show in Fig. 9(a) that where n is very small the network acts as a single network, of course. However, when we increase n more and more, at some point the wavy pattern appears and becomes more and more clear. In Appendix A 5, we find analytically a criterion for the emergence of a wavy distribution in a network of two modules. In Fig. 9(b) we show by changing the outer average degree, k_{out}, how the sparsity of the outer network affects the waviness of the distribution.

IV. DISCUSSION

In this paper we develop a framework to find analytically the distance distribution within networks with extreme community structure given the distributions of the inner and outer networks. We study here a model where we assume there is at most a single inter-link between modules. We showed that the SPL distribution has a wavy pattern in good agreement with simulations. Future work can investigate the validation of this model for real networks, where multiple links between modules exist.

ACKNOWLEDGMENTS

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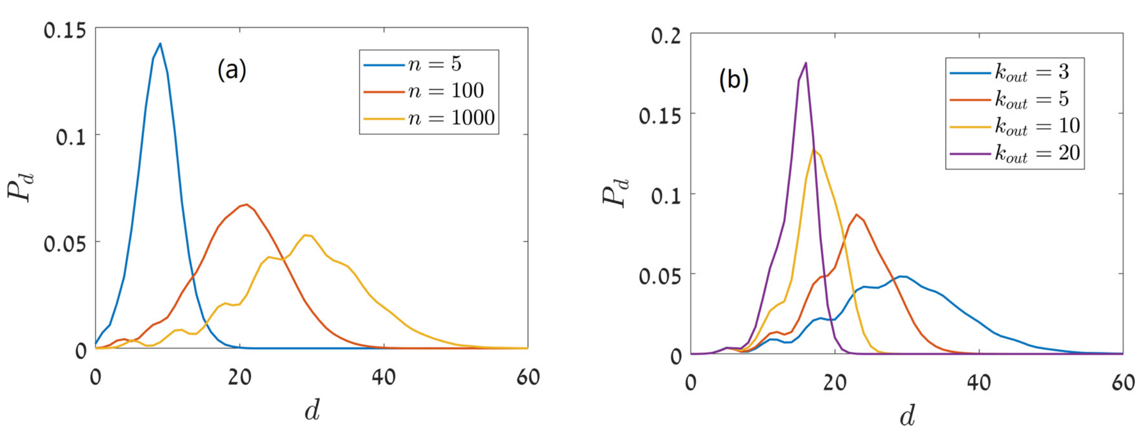

FIG. 9. Impact of the modules size and the outer degree on the wavy distribution. In (a) the network parameters are Out: ER, M=100, k=3, $w = 1$. In: ER, $k = 4$ and n changes. The results were averaged over five realizations of simulation. In (b) all the parameters are the same, except $n = 1000$ and k_{out} changes.

APPENDIX A: SPECIFIC NETWORKS

1. 1D lattice

Consider a 1D lattice with periodic boundaries with size L, where L is odd, for simplicity. The distances frequency of each node from all other nodes is given by

$$N_l = \begin{cases} 1, & l = 0 \\ 2, & 1 \leqslant l \leqslant (L-1)/2, \\ 0, & l > L/2 \end{cases} \tag{A1}$$

where N_l is the number of nodes in distance l from the source node. Then, the distance distribution, P_l, is obtained by

$$P_l = N_l/L. \tag{A2}$$

The generating functions of N_l and P_l satisfy

$$\begin{aligned} G_N(x) &= -1 + 2(1 + x + \cdots + x^{(L-1)/2}) \\ &= -1 + 2\frac{1 - x^{(L+1)/2}}{1-x}, \\ G_P(x) &= G_N(x)/L. \end{aligned} \tag{A3}$$

Thus,

$$G_{1D}(x) = G_P(x) = \frac{1}{L}\left(-1 + 2\frac{1 - x^{(L+1)/2}}{1-x}\right). \tag{A4}$$

Comment: In Eq. (A1) we counted twice the distances between different nodes i and j [(i, j) and (j, i)], and only once the distance (0) between a node i to itself. The reason is that we define P_l as the probability of the distance between two random nodes to be l. Indeed the probability to choose different nodes i and j is twice as large as the probability to choose the same node i twice. Note, it matters only for the value of P_0.

2. 2D lattice

Consider a 2D square lattice with periodic boundaries and size $L \times L$. For simplicity, we assume that L is odd. Then, the distances of each node from all other nodes have the following frequency:

$$N_l = \begin{cases} 1, & l = 0 \\ 4l, & 1 \leqslant l < L/2 \\ 4(L-l), & L/2 < l \leqslant L, \\ 0, & l > L \end{cases} \tag{A5}$$

and the distance distribution is

$$P_l = N_l/L^2. \tag{A6}$$

We note that N_l is obtained by a convolution of the series a_l and b_l, where

$$a_l = \begin{cases} 1, & 0 \leqslant l < L/2 - 1 \\ 0, & l > L/2 - 1 \end{cases}, \quad b_l = \begin{cases} 1, & 0 \leqslant l < L/2 \\ 0, & l > L/2 \end{cases}, \tag{A7}$$

such that

$$\begin{cases} N_0 = 1 \\ N_{l+1} = 4(a_l * b_l) \end{cases}. \tag{A8}$$

As a result, the generating functions of these sequences $(a_l,\ b_l,\ N_l,\ P_l)$ satisfy

$$\begin{aligned} G_N(x) &= 1 + 4xG_a(x)G_b(x) \\ G_P(x) &= G_N(x)/L^2. \end{aligned} \tag{A9}$$

But note that

$$\begin{aligned} G_a(x) &= 1 + x + \cdots + x^{(L-3)/2} = \frac{1 - x^{(L-1)/2}}{1-x}, \\ G_b(x) &= 1 + x + \cdots + x^{(L-1)/2} = \frac{1 - x^{(L+1)/2}}{1-x}. \end{aligned} \tag{A10}$$

Therefore, we obtain

$$\begin{aligned} G_{2D}(x) &= G_P(x) \\ &= \frac{1}{L^2}\left(1 + 4x\frac{(1 - x^{(L-1)/2})(1 - x^{(L+1)/2})}{(1-x)^2}\right). \end{aligned} \tag{A11}$$

3. 1D lattice of 2D lattices

Consider a circle of square lattices that are interconnected with one interlink between two random nodes from neighboring lattices. The interlinks have weight w while the intralinks have weight 1. Then, the distance distribution is obtained according to Eqs. (A4), (A11), and (4) by

$$G_{1D\times 2D}(x) = G_{2D}(x)G_{1D}(x^w G_{2D}(x)). \tag{A12}$$

This result is shown in Fig. 7(c).

4. Poisson distance distribution

To get insight of the wavy distribution, we assume here that the network has a Poissonian distance distribution. This will enable us to obtain analytically the DSPL in our model of extreme modular networks. Indeed random networks have in certain parameters range a distance distribution which can be approximated by a Poissonian distribution, as shown in Fig. 10. This changes with the degree and the network size significantly. For higher degrees it does not work so well, while for small degrees it does.

Thus, under proper conditions, if

$$P_l = \frac{\lambda^l}{l!}e^{-\lambda}, \tag{A13}$$

where $\lambda = \langle l \rangle$, then the generating function is as known

$$G(x) = e^{\lambda(x-1)}. \tag{A14}$$

Now, if both inner and outer networks have approximately Poisson distribution, then for P_d it is satisfied according to Eqs. (A14) and (4) that

$$G_d(x) = e^{\lambda_{\text{in}}(x-1)}e^{\lambda_{\text{out}}\left(x^w e^{\lambda_{\text{in}}(x-1)} - 1\right)}, \tag{A15}$$

where $\lambda_{\text{in}} = \langle l^{\text{in}} \rangle$ and $\lambda_{\text{out}} = \langle l^{\text{out}} \rangle$.

Still, it is difficult to find an explicit expression for the Taylor coefficients, which are P_d, in order to find some criteria for the emergence of wavy distribution. However, numerical calculation shows that if $\langle l^{\text{in}} \rangle$ is large enough relative to $\langle l^{\text{out}} \rangle$, then the wavy pattern appears. See Fig. 11.

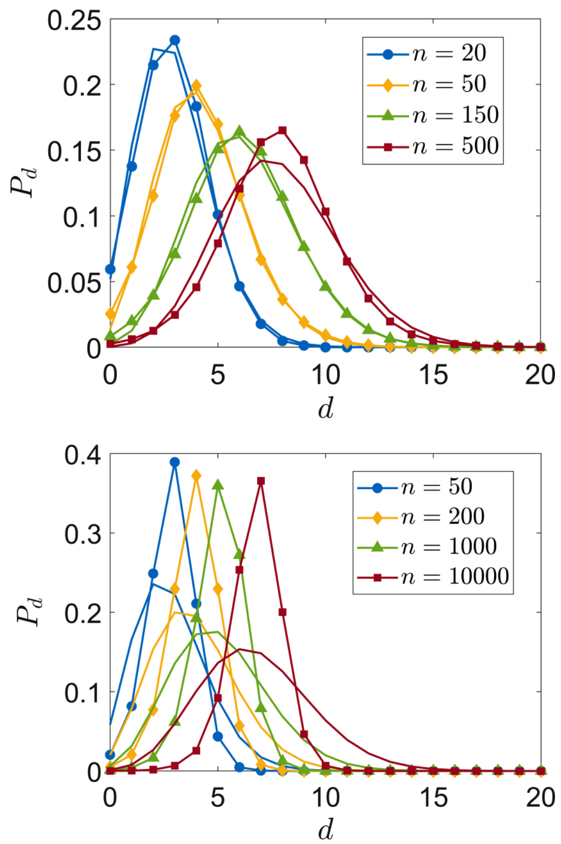

FIG. 10. Test of Poisson approximation for ER DSPL. Simulations were performed over ER network with $k = 2$ (top) and $k = 4$ (bottom). Marked lines represent simulation results and unmarked lines are Poissonian distributions with the same mean. For $k = 2$ the approximation is good, while as n increases further it becomes less accurate. However, when $k = 4$, there is a large deviation between theory result and simulations even for small system size. Generally, we see a difference that Poisson distribution has a standard deviation $\sqrt{\lambda}$ where the average is λ, while the DSPL does not change its standard deviation while changing its average for large n.

5. Two modules

To better understand the transition from a single peak to wavy distribution of distances, we study here a simple case which can be fully analyzed analytically. To this end,

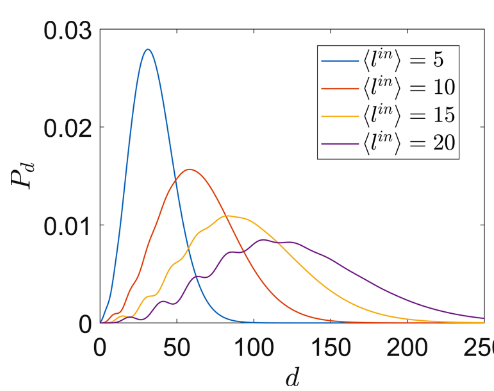

FIG. 11. Both inner and outer networks have Poisson distance distribution. Analytical results of Eqs. (A15) and (5), where $\langle l^{\text{out}}\rangle$=5.

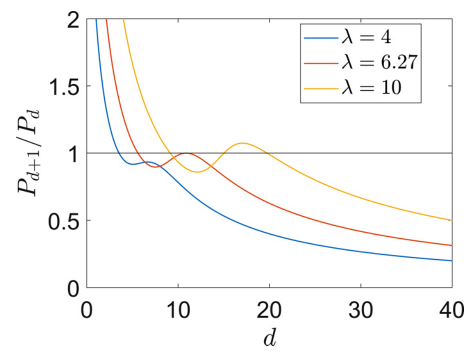

FIG. 12. Analysis of P_d, for two modules having Poissonian DSPL with average λ. The plot shows the ratio P_{d+1}/P_d according to Eq. (A19). If the ratio is greater than 1 then P_d increases, and if the ratio is lower than 1 then P_d decreases. In this analysis the interlinks weight $w = 1$.

we study a network of two connected nodes that satisfies $P_0 = 1/2$, $P_1 = 1/2$, and $P_l = 0$ for any other l. Hence, the generating function is

$$G_{\text{two}}(x) = \tfrac{1}{2}(1 + x). \quad \text{(A16)}$$

Let a network of two modules which have a Poisson DSPL. Then, according to Eqs. (A16), (A14), and (4) we obtain

$$G_{\text{two}\times\text{Poisson}}(x) = \tfrac{1}{2}e^{\lambda(x-1)}[1 + x^w e^{\lambda(x-1)}], \quad \text{(A17)}$$

where $\lambda = \langle l^{\text{in}}\rangle$.

Then we can find the coefficients of the Taylor series

$$G_{\text{two}\times\text{Poisson}}(x) = \frac{1}{2}\sum_{d=0}^{\infty}\frac{\lambda^d}{d!}e^{-\lambda}x^d + \frac{1}{2}\sum_{d=0}^{\infty}\frac{(2\lambda)^d}{d!}e^{-2\lambda}x^{d+w},$$

that yields

$$P_d = \begin{cases} \frac{1}{2}\frac{\lambda^d}{d!}e^{-\lambda}, & d < w \\ \frac{1}{2}\frac{\lambda^d}{d!}e^{-\lambda} + \frac{1}{2}\frac{(2\lambda)^{d-w}}{(d-w)!}e^{-2\lambda}, & d \geqslant w \end{cases}. \quad \text{(A18)}$$

If $w = 1$, then for $d \geqslant 1$

$$P_d = \frac{1}{2}\frac{\lambda^d}{d!}e^{-\lambda}\left(1 + \frac{d2^d}{2\lambda}e^{-\lambda}\right).$$

Next, we find the ratio

$$\phi(d,\lambda) := \frac{P_{d+1}}{P_d} = \frac{\lambda}{d+1}\frac{2\lambda + (d+1)2^{d+1}e^{-\lambda}}{2\lambda + d2^d e^{-\lambda}}. \quad \text{(A19)}$$

This ratio indicates whether the series P_d increases ($\phi > 1$) or decreases ($\phi < 1$). From Fig. 12 one can see that for small λ ($\lambda = 4$) P_d increases up to some value and then decreases. In contrast, for large λ ($\lambda = 10$) P_d increases again after decreasing, which indicates a wavy pattern. However, in the transition ($\lambda = 6.27$) two conditions are satisfied:

$$\begin{aligned} &\text{I} \quad \phi(d_c,\lambda_c) = 1, \\ &\text{II} \quad \frac{\partial\phi}{\partial d}(d_c,\lambda_c) = 0. \end{aligned} \quad \text{(A20)}$$

Numerical solution of these equations yields $\lambda_c \approx 6.27$. Namely, for two modules which have Poisson DSPL, if $\langle l^{\text{in}}\rangle > 6.27$, then two peaks will appear. Assuming each module is ER with $k_{\text{in}} = 2$ (see Fig. 10), we find numerically that the

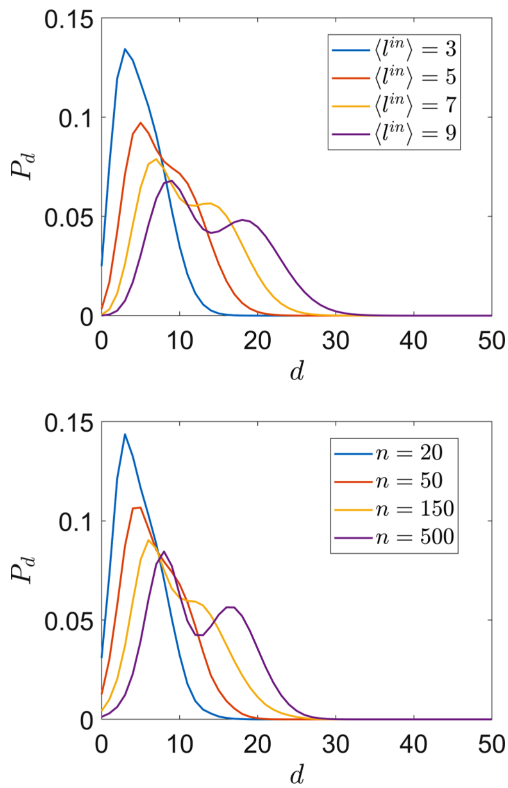

FIG. 13. Emergence of multiple peaks in two modules with Poisson DSPL. The upper panel is from theory [Eq. (A18)], and we see that the emergence of two peaks is for $5 < \langle l^{\text{in}} \rangle < 7$ which is in agreement with the value found $\lambda_c = 6.27$. In the lower panel we show simulations of ER with $k = 2$, and one can see that the transition is slightly above $n = 150$, consistent with the finding that approximately $n = 160$ gives $\langle l \rangle = 6.27$. It works well for $k = 2$ because for this range the Poisson approximation holds well as shown in Fig. 10.

required size should be approximately $n = 160$ in order to satisfy $\langle l \rangle > 6.27$. See Fig. 13 where the simulations results are consistent with this prediction.

For different values of w a similar analysis can be done. Higher values of w yield lower values of λ_c. As example, we find numerically that $\lambda_c \approx 3.31$, where $w = 2$. In contrast, where $w = 0$, then $\lambda_c \approx 8.38$.

6. Star graph

A star graph with n nodes has the following distance distribution:

$$N_l = \begin{cases} n, & l = 0 \\ 2(n-1), & l = 1 \\ 2\binom{n-1}{2}, & l = 2 \end{cases}, \tag{A21}$$

and $P_l = N_l/n^2$. Then

$$G_{\text{star}}(x) = P_0 + P_1 x + P_2 x^2. \tag{A22}$$

P_1 is about twice P_0, and P_2 is about n times P_0. Thus, for outer network star graph, if the inner network is such that there are separated peaks, there will be three peaks where the third one is much higher depending on n.

APPENDIX B: DIFFERENT CASES OF INTERLINKS CONNECTIONS

In this section, we analyze in detail the case of Sec. II C for which, when entering a module via an interconnected node i, we leave this module via different interconnected node j with probability p, or, when departing from the module via the same node with probability $1 - p$. See Fig. 5.

We denote l^{con} as the length of the path within the module which was taken during the course, excluding the first and the last modules. Thus, with probability $1 - p$, $l^{\text{con}} = 0$ (when entering and exiting were via the same node), and with probability p, $l^{\text{con}} = l^{\text{in}}$ (when entering and exiting the module has been done via different nodes). In the latter case, the distance between the two interconnected nodes is the typical random distance within the module. Therefore,

$$\begin{aligned} d &= l_1^{\text{in}} + l_2^{\text{in}} + w l^{\text{out}} + \sum_{i=1}^{l_{\text{out}}-1} l_i^{\text{con}} \\ &= l_1^{\text{in}} + l_2^{\text{in}} + w + \sum_{i=1}^{l_{\text{out}}-1} \left(l_i^{\text{con}} + w\right), \end{aligned} \tag{B1}$$

and if $l^{\text{out}} = 0$ then $d = l^{\text{in}}$. Hence,

$$\begin{aligned} G_d(x) &= G_{\text{out}}(0)G_{\text{in}}(x) + [G_{\text{in}}(x)]^2 x^w \\ &\quad \times ([G_{\text{out}}(x) - G_{\text{out}}(0)]/x) \circ (x^w(1 - p + pG_{\text{in}}(x))). \end{aligned}$$

As a result

$$\begin{aligned} G_d(x) &= G_{\text{out}}(0)G_{\text{in}}(x) \\ &\quad + [G_{\text{in}}(x)]^2 \frac{G_{\text{out}}(x^w(1 - p + pG_{\text{in}}(x))) - G_{\text{out}}(0)}{1 - p + pG_{\text{in}}(x)}. \end{aligned} \tag{B2}$$

Note that the last equation converges nicely to those of Secs. II A [Eq. (4)] and II B [Eq. (6)] at the limits $p = 1$ and $p = 0$, respectively.

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