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Robustness of interdependent networks based on bond percolation

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Abstract – Understanding the robustness of interdependent networks has attracted much attention in recent years. In many real scenarios, links may fail instead of nodes and how the interdependent networks behave in this case has not been adequately addressed. In this work, we investigate the link failures propagation mechanism for both two-layer and n-layer interdependent networks by using the self-consistent probabilities method which significantly simplifies the mathematical analysis of such systems. For bond percolation in which initial link failures occur in one layer, we find, analytically and via simulations, that the critical percolation threshold, p_c, of this system is lower than that of site percolation. Furthermore, for interdependent ER networks, in contrast to site percolation, bond percolation results show that p_c varies nonlinearly with the inverse of average degree. We also find, for the case of bond percolation where initial link failures occur in all layers, that the critical percolation threshold is the same as that of site percolation, but the behavior of the giant component above p_c is different. Our research brings insight to better understand the vulnerability of interdependent networks due to link failures.

Introduction. – In our real world, many infrastructure networks can be regarded as interdependent networks [1–5]. Understanding their robustness against cascading failures using percolation theory has attracted much attention in recent years [6–18]. These studies have shown that the behavior of interdependent networks is very different from that of single networks and significantly more vulnerable due to the dependency links. For studying robustness of interdependent networks, Buldyrev *et al.* [6] developed in 2010 a mathematical framework which is based on iterations of the generating function. Following this framework, many exciting results emerged. Parshani *et al.* [7] found that reducing the coupling interdependence strength leads to a change from a first (abrupt) to a continuous second-order percolation transition. Huang *et al.* [9] mapped the targeted-attacks problem to random attack and established the theoretical robustness analysis framework for the interdependent network under attacking high degree nodes. Shao *et al.* [10] introduced an analytical framework for the case of a number of support and dependency relationships. The authors of refs. [8,11–15,17] have studied how correlated patterns of interdependencies affect the robustness of interdependent networks. Liu *et al.* [18] recently studied a node-to-link failure propagation mechanism and weak interdependence across layers via a tolerance parameter. A novel phenomenon named mixed percolation transitions was found in this system. Yuan *et al.* [19] proposed and studied a generalized percolation model that introduces a fraction of reinforced nodes in the interdependent networks that can function and support their neighborhood. Wang *et al.* [20] have considered the situation where nodes usually cooperate and form groups to enhance their robustness to risks. More generally, Liu *et al.* [21] developed a theoretical framework based on generating functions and percolation theory to understand the breakdown and robustness of interdependent directed networks. Considering that many interdependent systems are spatially

embedded researchers recently expanded the idea of interdependent networks to a pair of spatially embedded networks [22–29]. They found that spatial interdependent networks are significantly more vulnerable.

In the above studies, many meaningful results have been obtained for percolation in the interdependent networks from different aspects. The common approach is that the initial failure, which leads to cascading failures and catastrophic events, occurs on nodes. However, in many realistic examples, the initial failure may occur on links instead of nodes. For instance, in the power grid system, the case of transmission line failure is quite realistic and occurs more frequently than shut down of a power station. In economy, breaking of a trade relationship is more common than a company bankruptcy. In airline systems, it is more likely to have a disruption of routes than the closure of airports. Considering this case, Hackett *et al.* [30] studied the robustness of multiplex networks based on bond percolation (BP). In this study, link failures could occur in any layer of the multiplex network. However, note that the authors considered as a measure of robustness the expected size of the giant connected component (GCC). This measure is useful for cases when both types of links can support each other such as two types of transportation systems, *e.g.*, subway and buses. However, when the two types of links represent different functions, for example, in transportation and power grid, one needs a more strict measure represented by the simultaneous existence of both the giant components in the multiplex, which is called the mutually connected giant components (MCGC) [16]. Although site and bond percolation are usually "equally" treated for single network because in the majority of the cases they belong to the same universality class [31], it could be more complicated for interdependent networks due to the dependency links and the diversity cases of bond percolation which could occur in one layer or all layers. How do the cascading failures propagate (due to link failures in one layer or in all layers of the interdependent network)? Indeed, to the best of our knowledge this problem has not been addressed properly. In this work, we model this phenomenon as a bond percolation happening in one layer or in all layers of the interdependent networks and study the robustness, via the condition of having a MCGC, of such a system.

The most common mathematical framework for studying the robustness of interdependent networks is based on iterations of the generating-function formalism. However, for bond percolation in the interdependent networks, it may lead to very complex formulations due to the mixing of node failures and link failures that happen together. We use a relatively simplified analytical framework known as the self-consistent approach [16,32–36] to analyze the critical behavior of interdependent networks for the case where the initial failure occurs on links in one layer or in all layers of the interdependent networks without going through the cascading process. We present here the detailed framework to study the nature of the critical phase transition, the value of the critical threshold, and the size of the MCGC for two-layer interdependent networks as well as for n-layer interdependent networks. We particularly focus on comparing the results of bond percolation occurring in one layer to that of the site percolation (SP) in the interdependent networks.

Cascading failure model. – The two-layer interdependent network consists of two networks, namely A and B, with degree distributions $P_A(k)$ and $P_B(k)$, respectively. Networks A and B both have N nodes, and each node in network A randomly depends on only one node in network B by a dependency link and vice versa. In our bond percolation model, the same as the case of site percolation, we define that a node can maintain its functionality if it is in its giant component network and its dependent node is also in the giant component of the other network, *i.e.*, it belongs to the mutually connected giant component (MCGC). The difference between site and bond percolation on the interdependent networks is that we randomly remove a fraction, $1-p$, of the internal links instead of nodes from the networks.

The general mathematical solution based on the self-consistent approach. – We firstly consider a single network A whose size goes to infinite and suppose the network is locally tree-like. Based on the self-consistent approach, we define two central quantities to get the basic self-consistent equation, the probabilities of finding a node in the giant component (GC) or a link leading to the GC [16,35]. We assume that x is the probability that a randomly selected link in network A leads to the giant component, thus $1-x$ is the probability that a randomly chosen link does not lead to the GC. We define p^A_{Lu} as the probability that the end node u of a randomly selected link L in network A belongs to the GC. To calculate p^A_{Lu}, we know the probability that following a link to find a node u which has a degree k is $\frac{P(k)k}{\langle k\rangle}$. The probability that this node is in the GC equals the probability that at least one of its other $k-1$ outgoing links must lead to the GC, which is $1-(1-x)^{k-1}$ (note that $(1-x)^{k-1}$ is the probability that all the $k-1$ outgoing links do not lead to the giant component). Thus,

$$p^A_{Lu} = \sum_k \frac{P_A(k)k}{\langle k_A\rangle}[1-(1-x)^{k-1}]. \tag{1}$$

From the definition of p^A_{Lu}, we know it has exactly the same meaning of x, thus, we can write a self-consistent equation for x:

$$x = p^A_{Lu} = \sum_k \frac{P_A(k)k}{\langle k_A\rangle}[1-(1-x)^{k-1}]. \tag{2}$$

Equation (2) is the very basic self-consistent equation for a single network.

Then we define p^A_u as the probability that a randomly chosen node u in network A belongs to the GC. For a

randomly selected node u, the probability that it is in the giant component equals the probability that at least one of its k links must lead to the giant component, which is $1-(1-x)^k$. Thus, we have

$$p_u^A = \sum_k P_A(k)[1-(1-x)^k], \tag{3}$$

where $P_A(k)$ is the probability that node u in network A has degree k. From the definition of p_u^A we know it is also the normalized size of the giant component u^∞. Thus,

$$u^\infty = p_u^A. \tag{4}$$

Based on eqs. (2) and (4), if we randomly remove a fraction, $1-p$, of links from network A (*i.e.*, there is a fraction of p links remaining), hence, out of the original probability, x, that a randomly selected link leads to GC, only a fraction of p links actually remains. The new self-consistent equation for x becomes:

$$x = p \cdot p_{Lu}^A. \tag{5}$$

The fact is that none of nodes is removed from the networks in bond percolation, unlike the cases of site percolation, the normalized size of the giant component u^∞ is also the same as eq. (4). However, for the case of site percolation, due to a fraction of p nodes remaining, we have

$$u^\infty = p \cdot p_u^A. \tag{6}$$

Substituting the result of x calculated from eq. (5) into eq. (4) for bond percolation or eq. (6) for site percolation, we can obtain the value of u^∞, namely, the fraction of nodes in the GC.

Now we consider two fully interdependent networks where we randomly remove a fraction, $1-p$, of links from network A, for studying the phase transition behavior. Following the above definition in single network, for two-layer interdependent networks, the MCGC is the steady state and is self-sustaining. Therefore, we define $x(y)$ as the probability that a randomly chosen link in network A (B) leads to the MCGC. To calculate the probability x, we should consider three constraints: first, the fraction remaining in the network A; second, the probability that the end node u of a randomly chosen link in A belongs to the GC, referred to as intra-layer condition; third, the probability that the node u's dependent node u', in network B is also in the GC, referred to as inter-layer condition. Only when both u and u' simultaneously belong to their GC, u and u' are in the defined MCGC and survive. Such event happens with probability $p_{Lu}^A \cdot p_{u'}^B$. Therefore, for calculating x, *i.e.*, the probability that a randomly chosen link in A leads to the MCGC, one can obtain

$$x = p_r^A \cdot p_{Lu}^A \cdot p_{u'}^B, \tag{7}$$

where p_r^A is the non-removed links in the network A, *i.e.*, the fraction remaining in the network A. If a fraction, $1-p$, of links is removed initially, then p_r^A equals p, otherwise p_r^A equals 1. According to eqs. (1), (3), we obtain

$$x = p \cdot \sum_k \frac{P_A(k)k}{\langle k_A \rangle}[1-(1-x)^{k-1}] \cdot \sum_{k'} P_B(k')[1-(1-y)^{k'}]. \tag{8}$$

Similarly, the probability that a randomly chosen link in B leads to the MCGC is

$$y = p_r^B \cdot p_{Lu'}^B \cdot p_u^A. \tag{9}$$

Note that if we initially do not remove any links from network B, so the first item p_r^B in eq. (9) is 1. If we also remove the fraction, $1-p$, of links from network B, p_r^B equals p. We will discuss this case later:

$$y = 1 \cdot \sum_k \frac{P_B(k)k}{\langle k_B \rangle}[1-(1-y)^{k-1}] \cdot \sum_{k'} P_A(k')[1-(1-x)^{k'}]. \tag{10}$$

Due to the one-to-one support and dependent relationship between network A and network B, the normalized size of the MCGC, namely u^∞, is the same for both networks. Because we did remove any node initially from the network A or B, the probability that a randomly selected node u in the MCGC is equal to the product of p_u^A and $p_{u'}^B$, which is

$$u^\infty = u_A^\infty = u_B^\infty = p_u^A \cdot p_{u'}^B. \tag{11}$$

According to eq. (3), we can obtain

$$u^\infty = u_A^\infty = u_B^\infty = \sum_k P_A(k)[1-(1-x)^k] \times \sum_{k'} P_B(k')[1-(1-y)^{k'}]. \tag{12}$$

In principle, eqs. (8) and (10) could be re-expressed as: $x = F_1(p,y)$ and $y = F_2(p,x)$. If a system has first-order phase transition, we know that at the critical point, $p = p_c$, the two functions $x = F_1(p,y)$ and $y = F_2(p,x)$ meet tangentially with each other, p_c satisfies the following equation:

$$\frac{\partial F_1(p_c,y)}{\partial y} \cdot \frac{\partial F_2(p_c,x)}{\partial x} = 1. \tag{13}$$

If u^∞ as a function of p shows a first-order phase transition, we can find the solution of p_c. By now, we derived the formulas of u^∞ and p_c. Solving the above eqs. (8), (10), (12), we can get the mutual giant component size u^∞, a key quantity indicating the extent of percolation. Solving eqs. (8), (10), (13), we can get the critical phase transition point p_c. We will consider the interdependent ER (Erdös and Rényi) networks which have a relative simple degree distribution as an example to perform the analytical solution for link failures in one layer and link failures in both layers of the interdependent networks.

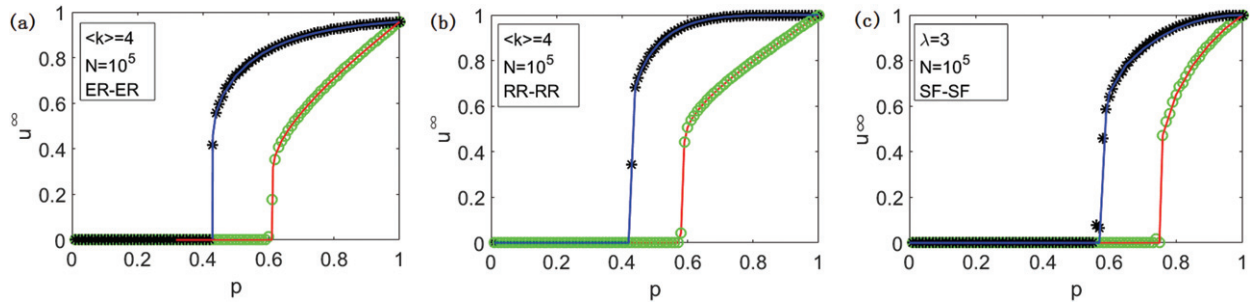

Fig. 1: Plot of u^{∞} as a function of p. The red line represents the theoretical result for the site percolation. The blue line represents the theoretical result for bond percolation. The green circles are the site percolation simulation results and the black stars are bond percolation simulation results. All the simulation results are obtained with 50 different realizations.

Bond percolation for removal in one layer. – We now apply our theoretical framework on the interdependent networks formed by ER networks and obtain the solutions of u^{∞} and p_c for link failures. We assume that the ER network A has an average degree $\langle k_A \rangle$ and the ER network B has an average degree $\langle k_B \rangle$, both of them follow a Poisson degree distribution:

$$P_A(k) = \frac{\langle k_A \rangle^k}{k!} \cdot e^{-\langle k_A \rangle}, \tag{14}$$

$$P_B(k) = \frac{\langle k_B \rangle^k}{k!} \cdot e^{-\langle k_B \rangle}. \tag{15}$$

Substituting eqs. (14), (15) into eqs. (8), (10), (12), we obtain:

$$x = p(1 - e^{-\langle k_A \rangle x})(1 - e^{-\langle k_B \rangle y}), \tag{16}$$

$$y = (1 - e^{-\langle k_B \rangle y})(1 - e^{-\langle k_A \rangle x}), \tag{17}$$

$$u^{\infty} = (1 - e^{-\langle k_B \rangle y})(1 - e^{-\langle k_A \rangle x}). \tag{18}$$

Solving eqs. (16), (17) and (18), we can get the relationship between u^{∞} and p, seen in fig. 1(a). In order to get the formula for p_c, we introduce a new variable, $\gamma = e^{-y}$, to reduce eqs. (16) and (17) into a single equation. To simplify the expression, we let $\langle k_A \rangle = a$, $\langle k_B \rangle = b$, so we can get:

$$\gamma = e^{-(1-\gamma^{ap})(1-\gamma^{b})}. \tag{19}$$

The solution of eq. (19) can be obtained numerically for any p. The critical point corresponds to the tangential condition,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\gamma} = \frac{\mathrm{d}}{\mathrm{d}\gamma}[e^{-(1-\gamma^{ap})(1-\gamma^{b})}], \tag{20}$$

or

$$1 = \gamma^{ap}[ap - (ap + b)\ \gamma^{b}] + b\gamma^{b}. \tag{21}$$

Here, the critical values of $\gamma = \gamma_c$ and $p = p_c$ satisfy the transcendental equations

$$\begin{cases} \gamma = e^{-(1-\gamma^{ap})(1-\gamma^{b})}, \\ 1 = \gamma^{ap}[ap - (ap + b)\ \gamma^{b}] + b\gamma^{b}. \end{cases} \tag{22}$$

Furthermore, if $a = b$, we can get

$$\begin{cases} p_c = \dfrac{1}{a}\dfrac{\ln(\frac{\ln \gamma_c}{1-{\gamma_c}^a} + 1)}{\ln \gamma_c}, \\ 1 = \left(\dfrac{\ln \gamma_c}{1-{\gamma_c}^a} + 1\right)\left[\dfrac{\ln(\frac{\ln \gamma_c}{1-{\gamma_c}^a} + 1)}{\ln \gamma_c} \cdot (1 - {\gamma_c}^a)\right] + a{\gamma_c}^a. \end{cases} \tag{23}$$

Equation (23) allows us obtain the value of p_c. By now, we have shown how to solve the values p_c and u^{∞}. In general, for other degree distributions, for example that of scale-free network, it is not feasible to get closed-form expressions. However, it can be solved using numerical solutions or graphical solutions demonstrated in fig. 2(c).

Results and discussion. – For easily comparing the case of link failures and the case of node failures, we can also get eqs. (24), (25) and (26) for x, y and u^{∞} using self-consistent probability method which agree with the original work [6,16]. These equations are slightly different from bond percolation,

$$x = p(1 - e^{-\langle k_A \rangle x})(1 - e^{-\langle k_B \rangle y}), \tag{24}$$

$$y = p(1 - e^{-\langle k_B \rangle y})(1 - e^{-\langle k_A \rangle x}), \tag{25}$$

$$u^{\infty} = p(1 - e^{-\langle k_B \rangle y})(1 - e^{-\langle k_A \rangle x}). \tag{26}$$

Intuition tells us that the critical point of bond percolation is smaller than the critical point of site percolation. It is hard to derive by formulas but easy to prove that by using graphical solution method. Comparing fig. 2(c) and fig. 2(b), the red curve representing eq. (17) does not change with p and only the blue curve representing eq. (16) moves toward right and down when p increases in fig. 2(c). So with small p, these two curves can tangentially meet each other. However, in fig. 2(b) the red curve always shows a lower value than that in fig. 2(a), which leads to a high value of p required to make two curves tangentially meet each other.

To verify our theoretical framework and the conclusion from the graphical solution method, we present both theoretical and simulation solutions for ER-ER networks, RR-RR networks and SF-SF networks with $\langle k_A \rangle = \langle k_B \rangle = 4$. Note that for RR-RR both networks A and B are random regular networks with $P_A(4) = P_B(4) = 1$. For

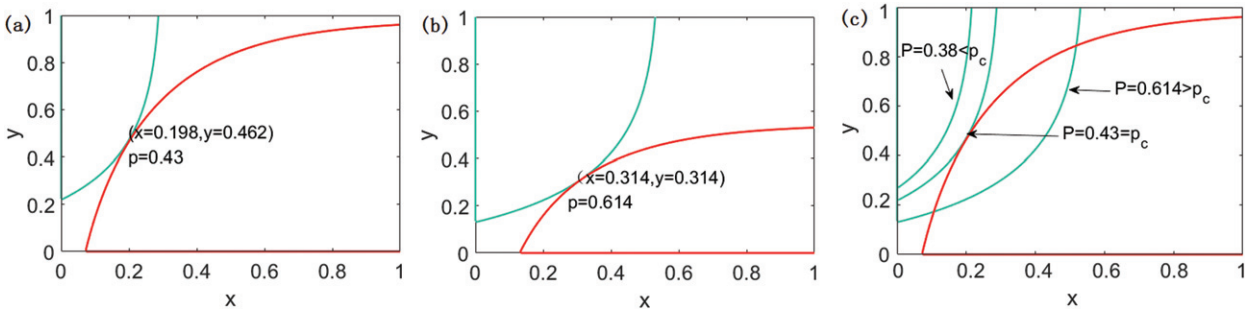

Fig. 2: (a) Bond percolation in the interdependent networks. The blue line represents eq. (16) and the red line represents eq. (17). At $p = 0.43$, the curves touch each other, where the slopes of the two curves are equal. It means that the percolation threshold is $p_c = 0.43$. (b) Site percolation in the interdependent networks. The blue line represents eq. (24) and the red line represents eq. (25). At $p = 0.614$, the curves touch each other, where the slopes of the two curves are equal. It means that the percolation threshold is $p_c = 0.614$. (c) Graphically solving the critical value of p_c for the case of bond percolation. The red line represents eq. (16) and the blue line represents eq. (17). At $p = p_c$, the curves touch each other, where the slopes of the two curves are equal. When $p > p_c$, there exist two intersection points and we know the higher one is valid whose coordinates correspond to the solution of x and y. When $p < p_c$, there is no intersection point which means the system collapses and a MCGC is absent.

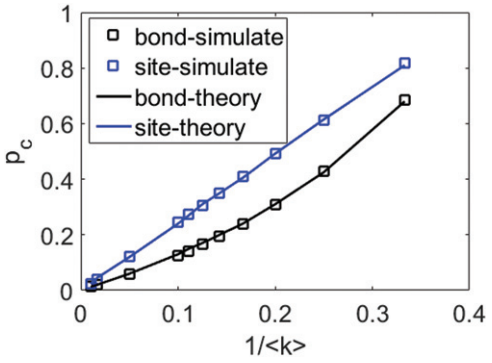

Fig. 3: p_c as a function of $\frac{1}{\langle k\rangle}$. Note the linearity for site and the nonlinearity for bond percolation.

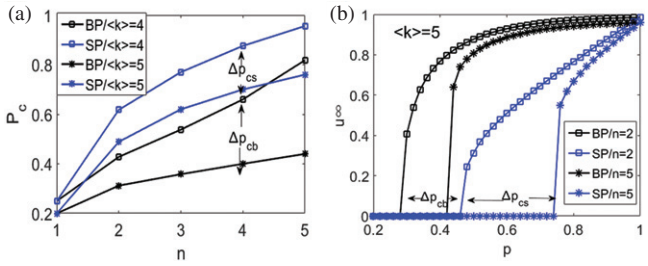

Fig. 4: Results of NON system. (a) Comparing p_c as a function of n for bond percolation and site percolation for two values of $\langle k\rangle$. (b) The expected size of MCGC, u^∞ as a function of p for $\langle k\rangle = 5$, $n = 5$ and $n = 2$.

SF-SF both networks A and B have the degree distribution of $P_A(k) = P_B(k) = \frac{\sum_{k_{\min}}^{k_{\max}} [(k+1)^{1-\lambda} - k^{1-\lambda}]}{(k_{\max}+1)^{1-\lambda} - (k_{\min}+1)^{1-\lambda}}$, where $k_{\min} = 2$, $k_{\max} = 150$, and the power law of this network is $\lambda = 3$. Substituting these formulas into eqs. (8), (10), (12), (13), the solutions could be found easily. Figure 1(a)–(c) shows good agreement between the theoretical and simulation results for the final giant component u^∞ as a function of p for ER-ER, RR-RR and SF-SF interdependent networks under random removal of $1-p$ nodes or links in one layer.

Figure 1 shows that the robustness of the network is higher for the case of link failures than in the case of node failures. However, the abrupt collapse (the failure cascade) is larger in the case of link failures than in the case of node failures. This can be understood from the fact that when links fail, although the nodes are connected, the average degree becomes smaller. At a critical point, the network abruptly collapses, due to a long cascade of failure.

Furthermore, results of critical percolation threshold for the ER-ER interdependent networks with different average k are shown in fig. 3. Surprisingly, in contrast to site percolation, the bond percolation results show that p_c varies nonlinearly with $\frac{1}{\langle k\rangle}$, as shown in fig. 3. Compared with site percolation, when the value of $\langle k\rangle$ is between 4 and 6, the distance between the two lines (blue line and black line) is larger than others, which means the robustness of interdependent network in this range is maximally different.

In addition, we study the the robustness of unbalance interdependent networks, *i.e.*, $\langle k_A\rangle \neq \langle k_B\rangle$. We find that the robustness of the interdependent network is stronger in the case of initial failure occurring in a network with higher average degree than occurring in a network with lower average degree.

Solutions for n-layer interdependent networks with one to one correspondence of dependent nodes. – Based on the above self-consistent probabilities method, we can easily develop an analytical framework for studying a network formed by n fully interdependent randomly connected networks, each composed of the same number of nodes N. We study the robustness of such a

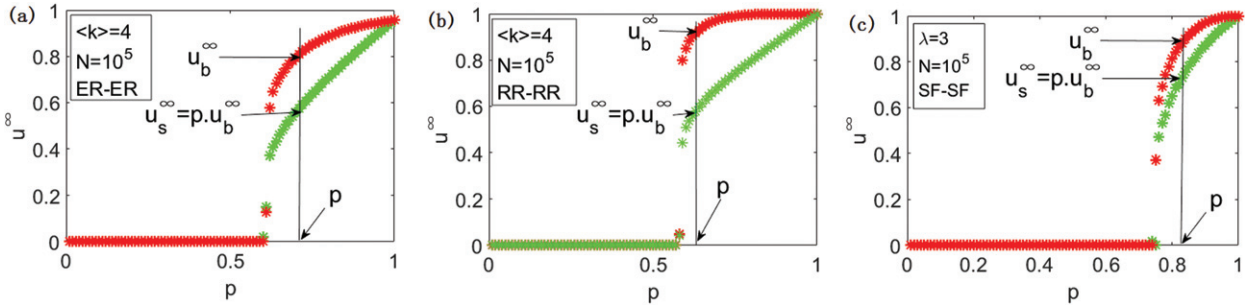

Fig. 5: Plot of u^∞ as a function of p. The green stars represent site percolation simulation results. The red stars are simulation results of bond percolation occurring in both layers of the interdependent networks. u_b^∞ represents the size of MCGC for bond percolation case and u_s^∞ represents the size of MCGC for site percolation case.

network of networks (NON) under link failures randomly occurring in one of the networks, after which a fraction p of its links survive. There are three types of loopless NON topologies, chain-like, star-like or tree-like. No matter the type of NON, the MCGC is the same due to the mutual dependence [37].

If a fraction $1-p$ of links are randomly removed from layer i, then it has a fraction p of links remaining in the layer after the attack. Extending from eq. (7), we obtain the probability that a randomly chosen link in network i leads to the MCGC as

$$x_i = p \cdot p^i_{Lu} \cdot \prod_{j \neq i} p^j_{u'}, \tag{27}$$

where $p^j_{u'}$ represents the probability that node u' of network j is in its GC, which has directly or indirectly dependent relationship with node u. Since no link is removed initially from layer $j(j \neq i)$, thus,

$$x_j = p^j_{Lu} \cdot \prod_{i \neq j} p^i_{u'}. \tag{28}$$

Similarly, since no node is initially removed, the probability that a randomly selected node is in the MCGC is

$$u^\infty = \prod_{i=1}^{n} p^i_u. \tag{29}$$

In a NON composed of n ER networks, we assume that the initial attacks occur in the first layer and all the n networks have the same average degree, that is $\langle k_1 \rangle = \langle k_2 \rangle = \cdots = \langle k_n \rangle = \langle k \rangle$, so we can get simplified equations,

$$x = p(1 - e^{-\langle k \rangle x})(1 - e^{-\langle k \rangle y})^{n-1}, \tag{30}$$

$$y = (1 - e^{-\langle k \rangle x})(1 - e^{-\langle k \rangle y})^{n-1}, \tag{31}$$

$$u^\infty = (1 - e^{-\langle k \rangle x})(1 - e^{-\langle k \rangle y})^{n-1}. \tag{32}$$

Similarly, solving eqs. (30)–(32), we can get p_c as a function of n and u^∞ as a function of p, as shown in fig. 4(a) and fig. 4(b), respectively. From fig. 4(a), we can see that in both bond percolation and site percolation the vulnerability significantly increases with n due to increasing cascading failures with n. We denote the critical percolation point for bond percolation as p_{cb} and for site percolation as p_{cs}. For the system with the same n but different k, Δp_{cb} is greater than Δp_{cs} and the difference between them is increasing with n, as shown in fig. 4(a). This result tells us that adding links to increase the average degree in order to improve the robustness of n interdependent networks for bond percolation is more meaningful than for site percolation. Figure 4(b) also shows that Δp_{cs} is larger than Δp_{cb} for the system with same $\langle k \rangle$ but different n, which means that vulnerability of system becomes higher with increasing of n for site percolation than for bond percolation.

Bond percolation for removal in all layers. – If we randomly remove a fraction, $1-p$, of links from network A and also from network B, the first term in eq. (10) should be p. In this case we know the self-consistent equations about x and y are the same as the site percolation. Therefore, we can see that when bond percolation occurs in all layers of interdependent networks, the system has the same critical value of p_c as in site percolation. However, due to the fact that all nodes remain in the network for bond percolation, there exist different giant components from the site percolation, *i.e.*, $u_s^\infty = p \cdot u_b^\infty$. The subscripts s and b refer to site percolation and bond percolation, respectively. In order to verify this, we also perform computer simulations. Results are shown in fig. 5. From fig. 5, we can see that for two cases (site percolation and bond percolation) the systems are collapsed at the same time. The size of the MCGC for site percolation is p times that for bond percolation due to the initially removed nodes. Results are agree well with our theory. From the above theoretical analysis, we can deduce that this conclusion is suitable for n-layer systems.

Conclusion. – In this work, we explored the link failure propagation mechanism by using the simplified self-consistent probabilities method. Our mathematical framework can significantly simplify the mathematical analysis of these systems compared to the generating functions method. We get two main interesting conclusions. First of all, for bond percolation occurring in one layer, we find, analytically and via simulations, that the critical

percolation threshold of this system is usually smaller than that of site percolation. This result is different from a single random network cases where the same critical threshold is found for both bond percolation and site percolation [16]. In contrast to site percolation on ER-ER interdependent networks, the bond percolation results show that p_c varies nonlinearly with the inverse of average degree. Secondly, for the case of bond percolation occurring on all layers of the interdependent networks, we find that the critical percolation threshold is the same with that of site percolation, however the behavior of the giant component as a function of p is different. This result agrees with the result of single random networks. Besides, when comparing the results of the bond percolation occurring in one layer to that of site percolation, we find that while the networks are more robustness, the collapse is more serious in the case of bond percolation than in the case of site percolation. In the NON system, even though the vulnerability increases with n for bond percolation in one layer, the trend is more gentle than in the case of site percolation. To summarize, the present study brings insight to better understand the vulnerability of interdependent networks due to link failures.

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