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PAPER

Possible origin for the similar phase transitions in k-core and interdependent networks

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Abstract

The models of k-core percolation and interdependent networks (IN) have been extensively studied in their respective fields. A recent study has revealed that they share several common critical exponents. However, several newly discovered exponents in IN have not been explored in k-core percolation, and the origin of the similarity still remains unclear. Thus, in this paper, by considering k-core percolation on random networks, we first verify that the two newly discovered exponents (fractal fluctuation dimension, d_f', and correlation *length* exponent, ν') observed in d-dimensional IN spatial networks also exist with the same values in k-core percolation. That is, the fractality of the k-core giant component fluctuations is manifested by a fractal fluctuation dimension, $\widetilde{d}_f = 3/4$, within a correlation *size* N' that scales as $N' \propto (p - p_c)^{-\widetilde{\nu}}$, with $\widetilde{\nu} = 2$. Here we define, $\widetilde{\nu} \equiv d \cdot \nu'$ and $\widetilde{d}_f \equiv d_f'/d$. This implies that both models, IN and k-core, feature the same scaling behaviors with the same critical exponents, further reinforcing the similarity between the two models. Furthermore, we suggest that these two models are similar since both have two types of interactions: short-range (SR) connectivity links and long-range (LR) influences. In IN the LR are the influences of dependency links while in k-core we find here that for $k = 1$ and $k = 2$ the influences are SR and in contrast for $k \geqslant 3$ the influence is LR. In addition, analytical arguments for a universal hyper-scaling relation for the fractal fluctuation dimension of the k-core giant component and for IN as well as for any mixed-order transition are established. Our analysis enhances the comprehension of k-core percolation and supports the generalization of the concept of fractal fluctuations in mixed-order phase transitions.

1. Introduction

The k-core model has been studied extensively since its introduction in 1960 to explore the question of what is the minimal number of colors required for covering graphs [1–5]. The model has a significant impact on the field of network science, and has been used to describe real-world systems such as social networks [6], the Internet [7], ecological networks [8], transportation networks [9], influential spreaders [10], etc. The k-core model can facilitate a better understanding of the organizational structure and behavioral characteristics of systems, by identifying the hierarchy of node connection patterns as well as the most densely connected subgraphs within a network called nucleus [11, 12]. The model can also provide insights in revealing the robustness and vulnerability of networks under attacks or failures [13–15].

The k-core percolation, recognized as an important application of the k-core model, primarily investigates the phase transition phenomena and critical behavior in network breakdowns. This process is accomplished by theory and simulations of the failure and disintegration of a network. One can initiate the failure through random removal of nodes, and then iteratively prune the graph by eliminating all nodes with a degree below k until the largest k-core remains. A k-core is the giant component of the original graph, where every node in the giant component has at least a degree k. The phase transition phenomenon in k-core percolation is typically characterized by the appearance and disappearance of a giant k-core component of the order of the size of the original network. Researches have shown that k-core percolation undergoes a mixed-order phase transition for $k \geqslant 3$, characterized by an abrupt jump in the order parameter (giant component) that resembles a first-order phase transition. However, it also exhibits scaling behaviors at and near the critical threshold, as typically observed in second-order phase transitions [13, 16, 17]. Further, considering k-core percolation for $k \geqslant 3$ in random networks, studies have revealed several critical exponents near the mixed-order phase transition threshold [18], containing the critical exponents $\beta_S = 1/2$, $\gamma_S = 1$, etc defined by

$$S(z) - S_c \propto (z - z_c)^{\beta_S}, \tag{1}$$

$$\chi_S = N\left(\langle S^2\rangle - \langle S\rangle^2\right) \propto (z - z_c)^{-\gamma_S}, \tag{2}$$

where $S(z)$ and S_c denote the fraction of the k-core giant component of a network with size N at mean degree z and at critical mean degree z_c, respectively; χ_S denotes the fluctuations of S; $\langle X\rangle$ represents the mean value of X. These exponents yield insights into the nature of critical phenomena that occur near phase transitions of the system, such as the scaling behavior near the critical transition.

In recent years, a model called interdependent networks (IN) has been developed and studied by Buldyrev *et al* [19]. This model has gained increasing attention [20–24], due to the growing interdependence between various systems in our modern world [25, 26]. The IN system is defined as two or more networks where pairs of nodes in different networks depend for functioning on each other. Thus, the failure of a node in one network triggers the failure of its dependent nodes in another network, as well as the failures of those nodes that are connected to the network via the failed nodes. Note that dependency links may also exist within a single network, yielding similar results [27, 28].

By studying IN, researchers can gain insights into the robustness and resilience of such interacting macroscopic systems, as well as exploring the mechanisms that drive cascading failures and collapse transition. In particular, studying the critical behavior of IN can help us in providing insights into the mechanisms yielding the critical phenomena and help to design more resilient interdependent systems. Noteworthy, research studies have shown that IN also undergo, like k-core [13, 14, 18], mixed-order phase transition [21, 29–31]. Moreover, in percolation of IN, the critical exponents β and γ also have respectively values equal to 1/2 and 1 like in k-core [13, 18, 32, 33], near the mixed-order phase transition point [21, 29, 34].

Zhou *et al* [35] studied the dynamics of cascading failures in IN and found that the mean value of the plateau time, τ (figure B3(a)), that is the number of iterations (time) until the system fully collapses at p_c scales as $\langle\tau_c\rangle \propto N^{1/3}$ and $\langle\tau_c\rangle \propto (p_c - p)^{-1/2}$. Very recently, Gross *et al* [36] have identified in IN critical characteristics of the order parameter fluctuations near the threshold of a mixed-order phase transition. They find that the fluctuations of the order parameter exhibit a fractal fluctuation dimension, d'_f, for length scales up to the correlation *length*. This is analogous to continuous second-order transitions, where near criticality the order parameter itself is a fractal. Moreover, Gross *et al* [36] found that for d-dimensional IN spatial networks the hyperscaling relation between d'_f, correlation *length* exponent ν' and β is valid, i.e.

$$d'_f = d - \beta/\nu', \tag{3}$$

for any d. Note that these two new exponents (i.e. d'_f and ν') also exist in interdependent random networks [36], like Erdős-Rényi (ER) networks [37].

These two models, k-core and IN, have flourished in their respective fields. Lee *et al* [18] have identified several shared behaviors and features between these two models. We claim here that in k-core percolation, like in IN, nodes fail due to two types of interactions that have different length scales. A node can fail because it does not belong to the giant component or because its degree is below k. In IN percolation, nodes are also removed based on becoming isolated from their own network or due to their dependencies on failing nodes in other networks or in the same network [27]. Thus, in both models, there exist two different types of interactions. In addition, percolation in both models exhibits mixed-order phase transition, with the same exponents $\beta = 1/2$ as well as the same scaling of the plateau times $\langle\tau_c\rangle \propto N^{1/3}$ [18, 29, 35] near criticality. Lee *et al* [24] revealed that the universal mechanism for mixed-order phase transitions may be related to long-range (LR) loops [24]. The two models indeed exhibit some similarities, but it remains unclear whether the newly discovered exponents for correlation *length* and fractal fluctuations found for IN [36] also exist in k-core percolation with the same values. Furthermore, the origin of the similarity in the phase transitions of the two models is also unclear.

Thus, in the present work, we explore these two questions, the similarity between the models and the mechanisms behind, by investigating k-core percolation in ER random networks. Specifically, using the node occupancy p as the controlled variable and the k-core giant component size M as the order parameter, we examine, like in [36], the fluctuations of the critical threshold and of the order parameter, as a function of N and $p-p_c$. Interestingly, we find here that k-core percolation also exhibits a fractal fluctuation dimension $\widetilde{d}_{\mathrm{f}} \equiv d_{\mathrm{f}}'/d = 3/4$ for network sizes up to the correlation *size* N', where $N' \propto (p-p_c)^{-\widetilde{\nu}}$ with $\widetilde{\nu} \equiv d\nu' = 2$. Note that $\widetilde{\nu}$ ($\widetilde{d}_{\mathrm{f}}$) is obtained by the scaling of the fluctuations of critical threshold, p_c (critical k-core giant component size, M_c) with respect to the network size N. Table 1 summarizes the critical exponents of k-core percolation and IN percolation in random networks. The identical critical exponents shared by both models, as presented in table 1 provide strong support for the similarity between the k-core and IN models. It is very plausible that the origin for the similarity between both systems (and probably in many other systems), lies in the similar fundamental mechanisms of these models. Specifically, the presence of two types of interactions: one is short range (SR) that is a node fail when becoming disconnected from the network and the other is LR influence by having a degree below k (in k-core) or not being supported via a dependency link (in IN). As shown below, the distribution of the distances caused by a failure in different k-core giant components provides strong support for this hypothesis. Our study contributes to a deeper understanding of the mechanisms of the k-core percolation, and also supports the generalization of the fractal fluctuations phenomena as well as the diverging correlation *length* and their meaning in general mixed-order phase transitions.

2. Possible origin for the similarity of phase transition between k-core model and IN

In this section, we will explore the critical behavior of k-core percolation in section 2.1 and of IN in section 2.2, both for ER random networks of size N with a mean degree z. We will also further try to uncover the origin of the similarity between both systems.

2.1. SR and LR interactions in k-core percolation

The k-core giant component is the maximal subgraph of the original graph, where every node in the k-core giant component has at least a degree k, as shown in figure 1(a). To obtain the k-core giant component of the original network, we first remove randomly a fraction of $1-p$ nodes from the original network. We now iteratively remove nodes, where at each iteration all nodes with a degree below k are removed until there are no such nodes. If a k-core giant component of the order of N nodes remains, we are above the transition p_c and if such a giant component does not exist, we are below p_c.

To test the range of influence, we randomly remove a node from the k-core giant component, where the giant component is obtained at node occupation rate $p(k)$, and $p(k)$ is close and above the critical threshold $p_c(k)$ such that $\Delta p = p(k) - p_c(k) = 0.05$. This removal will cause some nodes in the k-core giant component to have a degree below k, thereby triggering the removal of further nodes. This process will generate an avalanche and we will test here, how far, that is how many layers, in the network are influenced by the removed node. By analyzing this distribution, we expect to distinguish between SR and LR interactions. We plot in figures 1(b) and (c) the probability density function (PDF) of the relative maximal distance affected by the avalanche of the removed node in the k-core giant component. The relative distance of the initially removed node is defined as the ratio between its propagation distance and its maximum potential propagation distance, ranging from 0 to 1. The propagation distance of the initially removed node represents the maximum shortest path between subsequently removed nodes and the initially removed node, while the maximum possible propagation distance corresponds to the maximum shortest path between the nodes in the k-core giant component and the initially removed node. Small ratios that decrease with network size correspond to SR interactions and large ratios which do not change with network size, correspond to LR interactions. Note that, for k-core percolation, SR interaction refers to node failures triggering localized or limited-range cascading failures. The range is measured in shortest path distances. The ratios of relative distance distribution (with respect to the diameter) of such cascading failures is found to decrease with the network size. Conversely, LR interactions lead to cascading failures over long distances, and the ratios of relative distance distribution of these cascading failures are found unchanged with varying network sizes. In IN (as we show later), SR dependencies make it unlikely for node failures to propagate globally, while LR dependencies result in the propagation of failures over network scale distances. Note that, interaction between nodes is also a focus in higher-order network [38, 39]. We can clearly see in figures 1(b) and (c) that for $k=1$ and $k=2$, the process features SR interactions. However, for $k \geqslant 3$, all lines collapse together and behave as LR interactions. Moreover, the linearity in the log-linear plot suggests that the distribution follows an exponential distribution $P(\ell) \sim e^{-\lambda\ell}$ of distances ℓ. The slopes λ reveal that the decay for small k ($k=1$ and $k=2$) is faster than that of larger k ($k \geqslant 3$). Note that, the decay for $k=1$ is also faster than that of $k=2$.

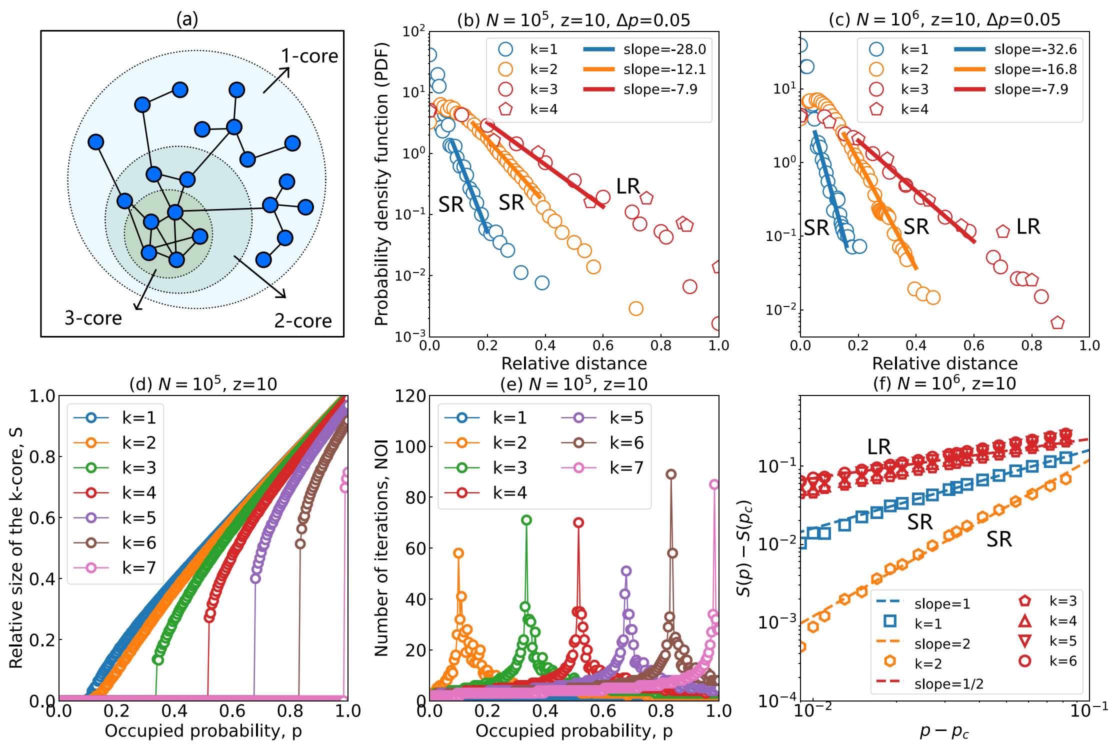

Figure 1. Results for k-core percolation. (a) Demonstration of the k-core giant component. The subgraphs surrounded by circles of different colors represent distinct cores. (b) probability density function (PDF) of the relative distance of failures caused by the removal of single nodes in the k-core giant component. The k-core giant component is obtained from the original ER network of network size $N=10^5$ and mean degree $z=10$ at node occupied fraction p near critical threshold p_c, satisfying $\Delta p = p - p_c = 0.05$. (c) Similar to (b) for the case $N=10^6$. We can clearly see in (b) and (c) for $k=1$ and $k=2$, only short-range (SR) interactions exist. This is since in these cases, the slope increases when N increases. However, for $k \geqslant 3$, the slope remains the same, meaning that long-range (LR) interactions exist. (d) The fraction of k-core giant component, S , as a function of the occupied fraction p of nodes for different k in ER network of size $N=10^5$ and mean degree $z=10$. (e) Number of iterations (NOI) to reach the steady state for different p. The critical threshold of the abrupt first order phase transition can be identified by the p at which NOI reaches maximum (as found for IN [40]). (f) The scaling law $S(p) - S(p_c) \propto (p-p_c)^\beta$ exists near the threshold p_c for different k. We have continuous transitions with $\beta=1$ for $k=1$ (SR interactions), and $\beta=2$ for $k=2$ (SR interactions, but a slightly longer than $k=1$) and $\beta=1/2$ for $k \geqslant 3$ (LR interactions), as found also in [13, 18]. Here we use a single realization of ER network of size $N=10^6$ and mean degree $z=10$.

Importantly, note that the slopes λ for larger k ($k \geqslant 3$) do not change with network size and thus supporting the hypothesis that the effect is LR. In contrast, for $k=1$ and $k=2$, the slopes increase (relative distances become smaller) with network size, suggesting they are of SR nature. As we will see later, in both systems k-core and IN, SR interactions yield a second order phase transition while LR interactions yield a mixed-order phase transition.

We also plot in figure 1(d) the fraction of the k-core giant component, S, in the k-core model with respect to the original network of mean degree z and size N, as a function of node occupation probability, p, for several values of k. Two distinct types of phase transitions can be observed. A second-order continuous phase transition can be seen for $k=1$ and $k=2$; and an abrupt transition is seen for $k \geqslant 3$, in agreement with earlier results [13]. For each curve in figure 1(d) except for $k=1$, we can identify the critical threshold p_c by identifying the p-value of the maximum number of iterations (NOI). This is since we expect that NOI diverges at p_c when N approaches infinite. Indeed, the values of p_c are in agreement with the theoretical values [13]. Note that this method of identifying p_c in k-core percolation is the same as that of identifying p_c in interdependent networks [40]. This method can help to identify p_c for systems such as spatial networks where a theory for p_c does not exist. Moreover, as shown in figure 1(f), for the k-core model, near the critical threshold, the scaling law $S(p) - S(p_c) \propto (p-p_c)^\beta$ exists, where $\beta=1$ holds for $k=1$ (like in regular percolation, corresponds to SR interactions); $\beta=2$ holds for $k=2$ (corresponds to SR interactions, figures 1(b) and (c), but somewhat longer than $k=1$); and $\beta=1/2$ holds for all $k \geqslant 3$ (see earlier results [13, 18]). Later we will show that very similar results also appear in IN.

2.2. SR and LR interactions in IN percolation

In this subsection, we consider IN where the dependency links are in the same network. As shown in figure 2(a), the single-layer network model incorporates both connectivity links (solid black lines) and

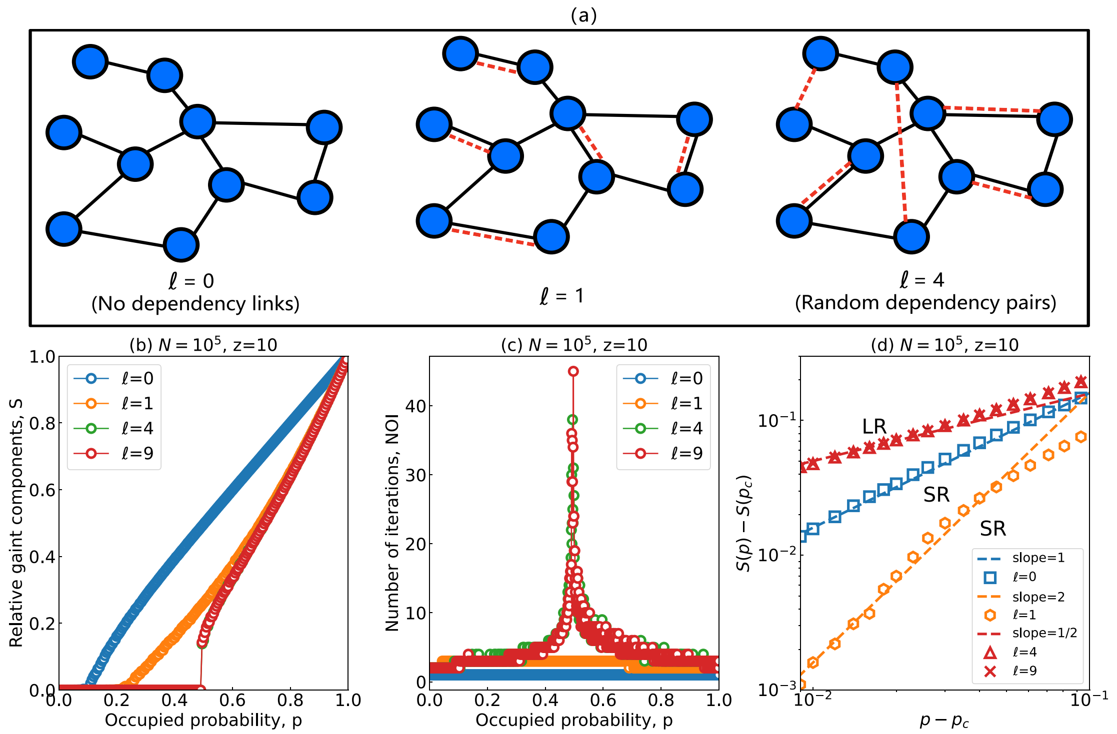

Figure 2. Results for percolation for the connectivity-dependency (CD) model. (a) Demonstration of the CD model, i.e. a single-layer network model that incorporates both connectivity links (solid black lines) and dependency links (dashed red lines), where the dependency links are subjected to a maximal shortest path length (ℓ) constraint. Small values of ℓ correspond to SR interactions, while large values of ℓ correspond to LR interactions. When $\ell \geqslant D$, where D is the network diameter (in this case, $D \cong 4$), the pairs of dependent nodes are chosen randomly. (b) The fraction of the giant component, S, as a function of the random occupied fraction p of nodes for different dependency length ℓ for ER random network of size $N=10^5$ and mean degree $z=10$. (c) Number of iterations (NOI) to reach the steady state for different p. The critical threshold of abrupt first order phase transition can be identified by the p at which NOI reaches maximum (as found for k-core (figure 1(e)) and for IN [40]). (d) The scaling law $S(p)-S(p_c) \propto (p-p_c)^\beta$ also exists near the threshold p_c for different distance ℓ. We have $\beta=1$ for $\ell=0$ (SR), and $\beta=2$ for $\ell=1$ (SR) and $\beta=1/2$ [21, 41] for $\ell=4,9$ (LR). Note that for $\ell=D$, Parshani *et al* [27] derived an analytical formula (see equation (A.1)), from which we can analytically derive the critical exponent $\beta=1/2$ (see appendix A for more details), supporting our results here. Note that for obtaining figure 2(d) we averaged over 100 realizations of ER network of size $N=10^5$ and mean degree $z=10$.

dependency links (dashed red lines), where the dependency links are subject to a maximal shortest path length (ℓ) constraint. For simplicity, we denote this model as the connectivity-dependency (CD) model. Each node i has one and only one dependency node j, with the constraint that the distance between the node i and its dependency node j is at most ℓ. Small values of ℓ correspond to SR interactions, while large values of ℓ of the order of the system diameter, correspond to LR interactions. When $\ell=0$, the single-layer network does not have dependency links, like regular percolation, and therefore $p_c=1/z$, see figure 2(b). When $\ell \geqslant D$, where D is the network diameter (in figure 2(a) case, $D \cong 4$), the pairs of dependent nodes are actually chosen randomly, like in the model of IN [19] or of a single network [28]. Note that the dependency links here are bidirectional, i.e. if a node i fails, it will cause the dependent node j to also fail, and vice versa.

Now we simulate the percolation process on the CD model. First, we construct the CD model upon an ER random network of size N with a mean degree z. In the first step, we randomly remove a fraction of $1-p$ of the nodes from the CD model. We remove the nodes iteratively while applying two processes. One is the percolation process: the failed nodes and their connectivity links are removed, causing other nodes to become disconnected from the network and fail. The other is the dependency process: the failed nodes trigger the failure of their dependent nodes, although these dependent nodes are still connected to the network via connectivity links. The iteration terminates when no such nodes are left. Similar to the k-core percolation, if a giant component of the order of N nodes remains, we are above the transition p_c and if such a giant component does not exist, we are below p_c.

Figure 2(b) depicts the fraction of the giant component, S, in the CD model with respect to the original network with mean degree z and size N, as a function of node occupation probability, p, for several values of ℓ. Like in k-core percolation (see figure 1(d)), two distinct types of phase transition can be observed. A second-order continuous phase transition is observed in figure 2(b) for $\ell=0$ and $\ell=1$, while an abrupt phase transition is observed for $\ell=4,9$. Notably, for $\ell=D \cong 9$, the results align with the findings in [27].

Table 1. Summary of critical exponents of k-core percolation and interdependent networks (IN) percolation of ER network.

Scaling	Exponent	k-core ($k \geqslant 3$)	Interdependent networks ($\ell = D$)[a]
$S(p) - S_c \propto (p - p_c)^{\beta}$	β	$\frac{1}{2}$ [13, 18, 32, 33]	$\frac{1}{2}$ [21, 41]
$\chi \propto (p - p_c)^{-\gamma}$	γ	1 [18]	1 [29, 34]
$S_c(N) - S_c(\infty) \propto N^{-\frac{\beta}{\bar{\nu}}}$ [a]	$\bar{\nu}$	2 [18, 32]	2 [29]
$N' \propto (p - p_c)^{-\widetilde{\nu}}$	$\widetilde{\nu}$	2 (equation (5))	2 [36]
$\sigma(M_c) \propto N^{\widetilde{d}_f}$	$\widetilde{d}_f$	$\frac{3}{4}$ (equation (8))	$\frac{3}{4}$ [36]
$\sigma(M) \propto (p - p_c)^{-\epsilon}$	ε	$\frac{1}{2}$ (equation (16))	$\frac{1}{2}$ [36]
$\langle \tau_c \rangle \propto N^{\psi}$	ψ	$\frac{1}{3}$ ([18], equation (B.1))	$\frac{1}{3}$ [29, 35]
$\langle \tau \rangle \propto (p - p_c)^{-\phi}$	ϕ	$\frac{1}{2}$ (equation (B.6))	$\frac{1}{2}$ [35]

[a] $S_c(N)$ denotes the fraction of the k-core giant component out of the original network of size N at p_c and $S_c(\infty)$ is for $N \to \infty$.

For each curve in figure 2(c) except for $\ell = 0, 1$, we can identify the critical threshold p_c by determining the p-value at the maximum number of iterations (NOI), like in the k-core model (see figure 1(e)) and like in the IN model [40]. Moreover, as shown in figure 2(d), near the critical threshold, the scaling law $S(p) - S(p_c) \propto (p - p_c)^{\beta}$ exists, where $\beta = 1$ holds for $\ell = 0$ (the limit of regular percolation); $\beta = 2$ is found for $\ell = 1$; and $\beta = 1/2$ holds for $\ell = 4, 9$.

Thus, we hypothesize that in both models, k-core and IN, the mechanisms behind the different phase transitions and therefore the outcome seem to be the same. In k-core percolation, the distribution of relative interaction distances indicates that: $k = 1$ and $k = 2$ correspond to SR interactions, while $k \geqslant 3$ corresponds to LR interactions (see figures 1(b) and (c)). Similarly, in the proposed CD model, we impose a restriction on the interaction distance ℓ of dependency links, where $\ell = 0$ and $\ell = 1$ represent SR interactions, and $\ell = 4, 9$ represents LR interactions. The behavior of the two models is similar for both the SR case, as well as LR case. Specifically, the comparison between figures 1(f) and 2(d) indicates that: $\beta = 1$ holds for both the CD model percolation ($\ell = 0$) and the k-core percolation ($k = 1$), corresponds to SR interactions; $\beta = 2$ is found for both the CD model percolation ($\ell = 1$) and the k-core percolation ($k = 2$), also associated with SR interactions, but with slightly longer interactions compared to $\beta = 1$. Furthermore, $\beta = 1/2$ is found for both the CD model percolation ($\ell = 4, 9$) and the k-core percolation ($k \geqslant 3$), which corresponds to LR interactions. The high consistency between the results and the critical exponents of the two models provides strong support for their similar fundamental mechanisms. Specifically, the similar behaviors observed in both models further support the hypothesis that both SR and LR influences are the origin of their similarity.

In the following sections, we aim to identify several new critical exponents of k-core giant component to further support the mechanisms behind the similarity between the two models. For a summary of all known exponents and all new exponents found here, see table 1.

3. Fractal fluctuations dimension and correlation *size* of k-core percolation

In this section, we study the critical exponents of the relation between the fluctuations of p_c and size N, which also represents how the correlation *size* scales with $p - p_c$. We further study how the fluctuations of M scale with N and $p - p_c$, i.e. what is the fractal fluctuations dimension.

3.1. Critical exponent of fluctuations of the critical threshold: correlation size

For a given network of size N we study in figure 3(a) the fluctuations of p_c, $\sigma(p_c)$, for different realizations scale as a function of N. The results suggest the following scaling with N,

$$\sigma(p_c) \propto N^{-1/\widetilde{\nu}},\ \widetilde{\nu} = 2. \tag{4}$$

Thus, equation (4) suggests that

$$N' \propto (p - p_c)^{-2}, \tag{5}$$

is the correlation *size* below which (for $N < N'$) the critical features exist and could be observed while above N' the critical regime disappears (see figure 4). Note that for lattices $\sigma(p_c)$ is measured as a function of L, the size of the lattice [36], i.e. $\sigma(p_c) \propto L^{-1/\nu'}$, suggesting correlation *length* $\xi' \propto (p - p_c)^{-\nu'}$. Here, in ER a linear length L does not exist, therefore we measure the fluctuations as a function of N, satisfying $N = L^d$, where d is the dimension of the spatial network. Thus, $\widetilde{\nu}$ can be regarded as a correlation *size* exponent, in analogy to the correlation *length* exponent ν' in d-dimensional spatial network, satisfying:

$$\widetilde{\nu} \equiv d \cdot \nu'. \tag{6}$$

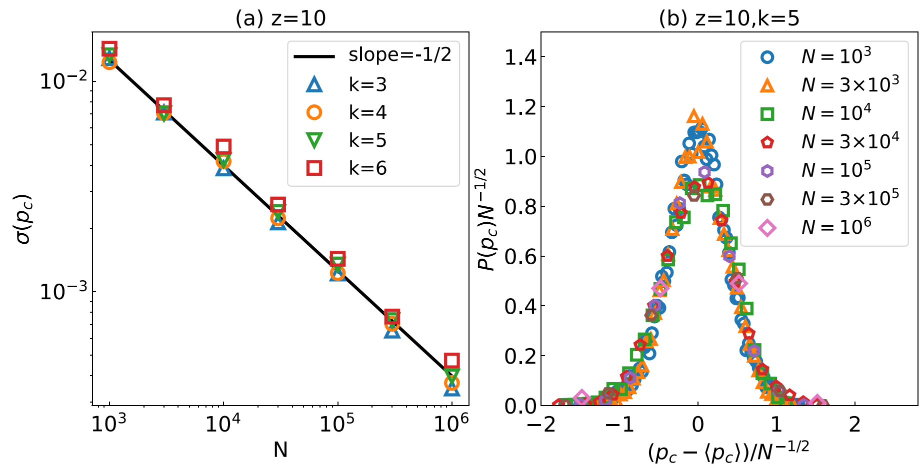

Figure 3. The correlation *size* exponent of k-core percolation in ER network. (a) The scaling relation between the standard deviation of the k-core percolation threshold, p_c, and network size N. It exhibits the same scaling relation as found for IN [36]. (b) The distribution of p_c follows the scaling relation of equation (7). Here we simulated 10^4 realizations for $N \leqslant 3 \cdot 10^5$ and 200 realizations for $N = 10^6$. Note that, we consider $k = 5$ as an example for the distribution shown here, but the same holds true for other values of $k \geqslant 3$. We have included the results for $k = 3$ in the appendix, which can be seen in figure B1(a).

In this study, we utilize finite-size scaling to measure it by following a similar approach given in [42–44].

The distribution of p_c for different realizations with respect to mean $\langle p_c \rangle$ for different N values is shown to behave in figure 3(b) as a scaling function,

$$P(p_c)\, N^{-1/\widetilde{\nu}} \sim F\left[(p_c - \langle p_c \rangle)/N^{-1/\widetilde{\nu}}\right], \tag{7}$$

where $F(x)$ can be well approximated by a Gaussian distribution. Here, all distribution curves collapse into a single curve by rescaling with the correlation *size* exponent $\widetilde{\nu}$. Note that the obtained exponent $\widetilde{\nu}$ as well as the scaling in equation (4) for k-core is the same as those found in the IN [36].

3.2. Fractal fluctuations dimension of the k-core giant component

For a given N we study in figure 4(a) the fluctuations of the k-core giant component M, $\sigma(M)$, at and near the threshold p_c based on different realizations, and then plot it as a function of N. The results at criticality suggest the following scaling with N,

$$\sigma(M_c) \propto N^{\widetilde{d}_{\mathrm{f}}},\ \widetilde{d}_{\mathrm{f}} = 3/4. \tag{8}$$

Here $\widetilde{d}_{\mathrm{f}}$ is the fractal fluctuations dimension of the k-core giant component in random networks with respect to the size N. While in d-dimensional lattices, $\sigma(M_c)$ is measured as a function of L, i.e. $\sigma(M_c) \propto L^{d_{\mathrm{f}}'}$ [36]. Since $N = L^d$, we have

$$\widetilde{d}_{\mathrm{f}} \equiv d_{\mathrm{f}}'/d. \tag{9}$$

Further, similar to the previous section, by rescaling with the fractal fluctuations dimension, $\widetilde{d}_{\mathrm{f}}$, we collapse in figure 4(b), the distribution of M_c for different N according to the scaling,

$$P(M_c)\, N^{3/4} \sim F\left[(M_c - \langle M_c \rangle)/N^{3/4}\right]. \tag{10}$$

While equation (8) is valid at the critical threshold, away from the critical threshold and above the correlation *size*, we can see the normal scaling, i.e.

$$\sigma(M) \sim N^{1/2}. \tag{11}$$

Close to the critical threshold, a crossover between these two behaviors is observed, which can be described using a scaling function $f(\mu)$ as

$$\sigma(M) \propto N^{3/4} f(\mu), \tag{12}$$

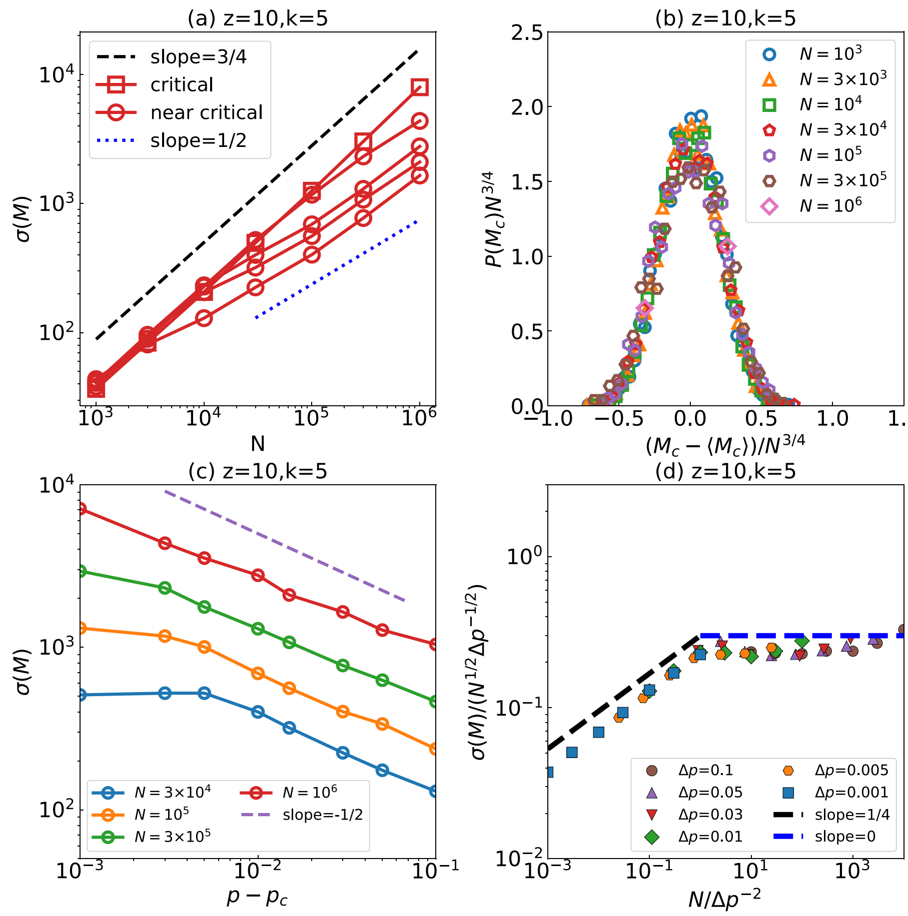

Figure 4. Fractal fluctuations dimension of the critical mass M_c of k-core percolation in ER network near p_c. (a) Scaling relation between the fluctuations of M and the network size N at and near the critical threshold p_c. Here we choose $\Delta p = p - p_c$ shown up to down being $\{0.003, 0.01, 0.015, 0.03\}$. The crossover in the slope represented by the exponent $\widetilde{d}_f$ is clearly seen. That is, within a correlation *size*, N' (equation (5)), fractality can be observed with $\widetilde{d}_f = 3/4$ while above the correlation *size*, there is no fractality and equation (11) appears. (b) The distribution of M_c follows the scaling relation in equation (10). (c) Scaling relation between the fluctuations of M and $p - p_c$. Note that, the scaling relation between the fluctuations of S and $p - p_c$ (see equation (2)) has been also investigated in [18]. $\sigma(M) \propto (p - p_c)^{-1/2}$ can be derived from equation (2), further supporting our findings here. (d) Fractal fluctuations behavior in the variation of the mass M near the critical mixed-order percolation threshold, supporting equation (16). Our results for k-core follow the same scaling function, crossover and critical exponents found in interdependent networks [36]. Note that, an analogous crossover has been also observed in [18], further supporting our fractality findings here. We considered here $k = 5$ as an example, but the same holds true for other values of $k \geqslant 3$ as can be seen in the appendix in figures B1(b) and B2.

with

$$\mu = (p - p_c)^{\alpha} \cdot N, \tag{13}$$

where $f(\mu)$ is a piecewise function satisfying $f(\mu) \propto$ constant for $\mu < 1$ and $f(\mu) \propto \mu^m = (p - p_c)^{\alpha m} \cdot N^m$ for $\mu > 1$. Thus, we have:

$$\sigma(M) \propto \begin{cases} N^{3/4} & \text{for } \mu < 1 \\ N^{3/4+m}(p - p_c)^{\alpha m} & \text{for } \mu > 1. \end{cases} \tag{14}$$

In figure 4(a) we can see that $\sigma(M) \propto N^{1/2}$ for $\mu > 1$, implying $3/4 + m = 1/2$, i.e. $m = -1/4$. Thus, we have:

$$\sigma(M) \propto \begin{cases} N^{3/4} & \text{for } \mu < 1 \\ N^{1/2}(p - p_c)^{-\alpha/4} & \text{for } \mu > 1. \end{cases} \tag{15}$$

In order to determine the value of α, we plot $\sigma(M)$ against $p - p_c$ in figure 4(c). We obtain $\sigma(M) \propto (p - p_c)^{-1/2}$, implying that $\alpha = 2$. Finally, to support equation (12) and the obtained value of α, we created a scaled plot shown in figure 4(d), depicting $\sigma(M)/\left(N^{1/2}(\Delta p)^{-1/2}\right)$ against $N(\Delta p)^2$. As can be seen, we achieve a satisfactory scaling collapse with $\alpha = 2$, i.e. we have:

$$\sigma(M) \propto \begin{cases} N^{3/4} & \text{for } N < N' \propto (p - p_c)^2 \\ N^{1/2}(p - p_c)^{-1/2} & \text{for } N > N'. \end{cases} \tag{16}$$

Thus, k-core features again the same universal scaling function as that of interdependent network [36].

4. Hyper-scaling relation for fluctuations in mixed-order transition

In this section, we establish analytical arguments for a universal hyper-scaling relation for the fractal fluctuations dimension of the order parameter in a mixed-order transition in spatial networks of size, $N = L^d$, which also provides insight for non-spatial random networks.

We start with the scaling of the fraction of k-core giant component, S, close to criticality for both second-order and mixed-order transitions,

$$S(p) - S_c \propto (p - p_c)^{\beta}. \tag{17}$$

Here, for second order transition $S_c = 0$ while for mixed order S_c is finite. Next, we substitute the size (number of sites) of the giant component, M_c, at p_c and $M(p)$ away (but closeby) from the critical threshold

$$\frac{M(p)}{L^d} - \frac{M_c}{L^d} \propto (p - p_c)^{\beta}. \tag{18}$$

Based on the general case, equation (18), one can obtain the well-known hyperscaling relation for continuous second-order transitions [42, 44]. Substituting $M_c = 0$ and $M(p) \propto L^{d_f}$ into equation (18), yield $L^{d_f}/L^d \propto (p - p_c)^{\beta}$ and since $\xi \propto (p - p_c)^{-\nu}$, the hyperscaling relation

$$d_f = d - \beta/\nu, \tag{19}$$

is derived.

Different from the continuous phase transition case, in the mixed-order phase transition case, for equation (18), we evaluate instead the standard deviation of both sides

$$\sigma\left(\frac{M(p)}{L^d} - \frac{M_c}{L^d}\right) \propto \sigma\left((p - p_c)^{\beta}\right). \tag{20}$$

Using the known relation $\sigma(X^a) \sim X^{a-1}\sigma(X)$ [45] and assuming that the dominant variations between different realizations are in p_c and M_c, we get

$$\frac{\sigma(M_c)}{L^d} \propto (p - p_c)^{\beta - 1}\sigma(p_c). \tag{21}$$

Assuming, $\sigma(M_c) \propto L^{d'_f}$, for $L < \xi'$, we get

$$\frac{L^{d'_f}}{L^d} \propto (p - p_c)^{\beta - 1}\sigma(p_c). \tag{22}$$

Substituting into equation (22) the scalings $\xi' \propto (p - p_c)^{-\nu'}$ and $\sigma(p_c) \propto L^{-1/\nu'}$, we obtain,

$$L^{d'_f - d} = \xi'^{-\frac{\beta - 1}{\nu'}} L^{-\frac{1}{\nu'}}. \tag{23}$$

Taking the limit $L \to \xi'$, we obtain,

$$\xi'^{d'_f - d} = \xi'^{-\frac{\beta}{\nu'}}. \tag{24}$$

Thus, deriving the new but analogous to equation (19) hyperscaling relation, equation (3), for the fractal fluctuations dimension, $d'_f = d - \beta/\nu'$.

Note that the hyperscaling relations given by equation (3) (for mixed-order phase transitions) and equation (19) (for second-order phase transitions) are similar, but the interpretation of the fractal dimensions d'_f and d_f are different. For second-order continuous phase transitions, d_f describes the fractal dimension of the bulk [42, 44], while for mixed-order phase transitions, d'_f describes the fractal dimension of the fluctuations of M_c, i.e. how $\sigma(M_c)$, scales with L. Below the correlation *length* the fluctuations are fractals, but above they behave normally. Note that the hyperscaling relation, equation (3) holds for spatial networks of different dimension d. Correspondingly, it also holds for random networks like the ER networks. Substituting equations (6) and (9) into equation (3), we can obtain the hyperscaling relation for random networks,

$$\widetilde{d}_{\mathrm{f}} = 1 - \beta/\widetilde{\nu}. \tag{25}$$

As expected, the evaluated critical exponents $\beta = 1/2$, $\widetilde{\nu} = 2$ and $\widetilde{d}_{\mathrm{f}} = 3/4$ found here satisfy this hyperscaling relation, equation (25). Noteworthy, the hyperscaling relation equation (3) with identical exponents as found here for k-core, is also valid for IN [36].

5. Discussions and summary

In the present study, we investigate systematically the k-core percolation phase transitions in ER random networks to uncover the similarities of percolation in the k-core and IN models, along with identifying the possible underlying mechanisms for the similarities. We examine several new critical exponents found recently in IN [36], such as the critical exponents of fluctuations of p_c, representing the diverging correlation *size*, N', and of the order parameters in different realizations of size, N, representing the fluctuation fractal dimension. The findings are summarized in table 1, where the common critical exponents of k-core percolation and IN reveal intriguing similarities between the two models.

Our results suggest that the similarity of the criticality between the two systems (and probably to many other systems) originates from the two types of interactions that exist in both systems, one is the connectivity links which are SR and the other is the k-core ($k \geqslant 3$) or the dependency (large ℓ) which are both LR. When plotting the distributions of distances caused by a failure in different cores, we found that for $k \geqslant 3$ the interactions are LR, while for $k = 1$ and $k = 2$ the interactions are SR (see figure 1). Similar to k-core percolation, percolation results of the CD model again show that SR interactions exist for $\ell = 0$ and $\ell = 1$ and thus yield a second order phase transition, while LR interactions, e.g. $\ell = D$ yields a mixed order phase transition. In both models (and potentially also for other models) the mechanisms behind the different phase transitions seem the same, and SR and LR influences are the origin. Moreover, we demonstrate that the fractal fluctuations dimension of the order parameter in the mixed-order phase transition satisfies the hyperscaling relation, $d'_{\mathrm{f}} = d - \beta/\nu'$.

Our study offers valuable insights into the striking similarities between k-core percolation and percolation in IN, shedding light on the phenomenon of fractal fluctuations dimension observed in mixed-order phase transitions. The investigation into the original mechanism of these similarities enhances our understanding of the intricate behaviors near the phase transition point, contributing to the establishment of a more rigorous and universally applicable theoretical foundation. These findings are poised to extend the interdisciplinary applications of both models, paving the way for a deeper understanding of breakdown of complex systems. They hold the potential to improve the resilience of real-world systems, such as communication networks, transportation networks, and energy networks. Furthermore, the insights gained have implications for engineering and technology fields, offering nuanced understanding for endeavors like materials design and manufacturing processes.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

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Appendix A. Critical exponent β in CD model for $\ell = D$

Here we consider the single-layer network model that incorporates both connectivity links and dependency links, where the dependency links are subject to a maximal shortest path length constraint ℓ called here the CD model, see figure 2(a). When $\ell = D$, where D is the network diameter, dependency pairs are randomly selected. Parshani *et al* [27] derived the formula:

$$S(p) = p^2 (1 - \exp(-zS(p)))^2 . \tag{A.1}$$

suggesting:

$$\exp(-zS) = 1 - S^{1/2}/p. \tag{A.2}$$

Here for simplicity, we denote $S(p)$ as S. To analytically find the critical exponent β, we define:

$$f(p,S) = S - p^2 (1 - \exp(-zS))^2 . \tag{A.3}$$

Letting $f(p,S) = 0$, equation (A.1) can be obtained. To find the critical exponent β near the critical point $(p_c,\ S_c)$, we can expand $f(p,S)$ near $p = p_c$ and $S = S_c$:

$$f(p,S) = f(p_c,S_c) + f'_S(p_c,S_c)(S - S_c) + f'_p(p_c,S_c)(p - p_c) + \frac{f''_{SS}(p_c,S_c)}{2!}(S - S_c)^2 + \frac{f''_{pp}(p_c,S_c)}{2!}(p - p_c)^2 + \frac{f''_{Sp}(p_c,S_c) + f''_{pS}(p_c,S_c)}{2!}(S - S_c)(p - p_c) + \ldots = 0. \tag{A.4}$$

Here, f'_S and f'_p denotes the partial derivatives of f with respect to S and p; f''_{SS} and f''_{pp} denotes the second order partial derivatives of f with respect to S and p; f''_{Sp} and f''_{pS} denotes the second mixed partial derivatives of f. At the critical point $(p_c,\ S_c)$, calculating f'_S, f'_p and f''_{SS} and substituting into equation (A.2), we obtain

$$f'_S(p_c,S_c) = 2zS_c - 2zp_cS_c^{1/2} + 1 = 0 \tag{A.5}$$

$$f'_p(p_c,S_c) = -2S_c/p_c \neq 0 \tag{A.6}$$

$$f''_{SS}(p_c,S_c) = -zp_cS_c^{-1/2} + 2z \neq 0. \tag{A.7}$$

Here, $f'_S(p_c,S_c) = 0$ is the condition for the first-order abrupt phase transition [27], and we know that equation (A.1) yields first-order abrupt phase transition. $f'_p(p_c,S_c) \neq 0$ is because $S_c \neq 0$. $f''_{SS}(p_c,S_c) \neq 0$ holds, since if it is not true, it will lead to a contradiction with equation (A.5). Then rearranging equation (A.4) yields

$$(S - S_c)^2 = -2\frac{f'_p(p_c,S_c)}{f''_{SS}(p_c,S_c)}(p - p_c) - 2\frac{f''_{Sp}(p_c,S_c) + f''_{pS}(p_c,S_c)}{f''_{SS}(p_c,S_c)}(S - S_c)(p - p_c) + \ldots \tag{A.8}$$

Here, since the first term in equation (A.8) is the leading term, we derive the scaling relation $S - S_c \propto (p - p_c)^{1/2}$, i.e. $\beta = 1/2$, supporting figure 2(d).

Appendix B. Critical exponent in k-core percolation

B.1. The collapsed PDF in k-core percolation

Here for $k = 3$ core percolation in ER network, we collapse the PDF of critical threshold p_c (critical mass M_c) together utilizing the same scaling relationship given in equation (7) (equation (10)), as shown in figure B1.

B.2. Fractal fluctuations of the k-core giant component

For $k = 3$ core percolation in ER network, the scaling behavior of fluctuation of k-core giant component at and near threshold is plotted in figure B2, which shows the same critical behavior as for $k = 5$ shown in figure 3, section 3.2. Note that it is the same scaling function found for percolation of IN with the same exponents [36].

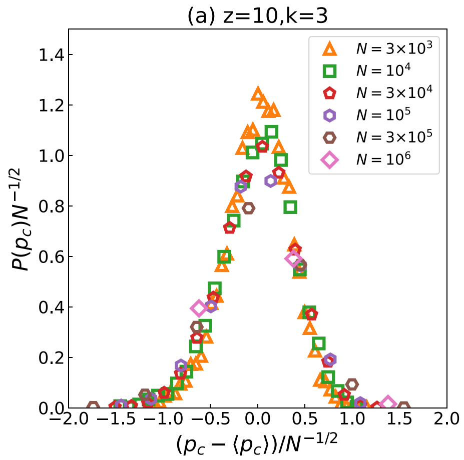

Figure B1. The collapsed Probability Density Function (PDF). (a) The distribution of p_c follows the scaling relation in equation (7). (b) The distribution of M_c follows the scaling relation in equation (10). It exhibits the same scaling relation as in interdependent networks [36].

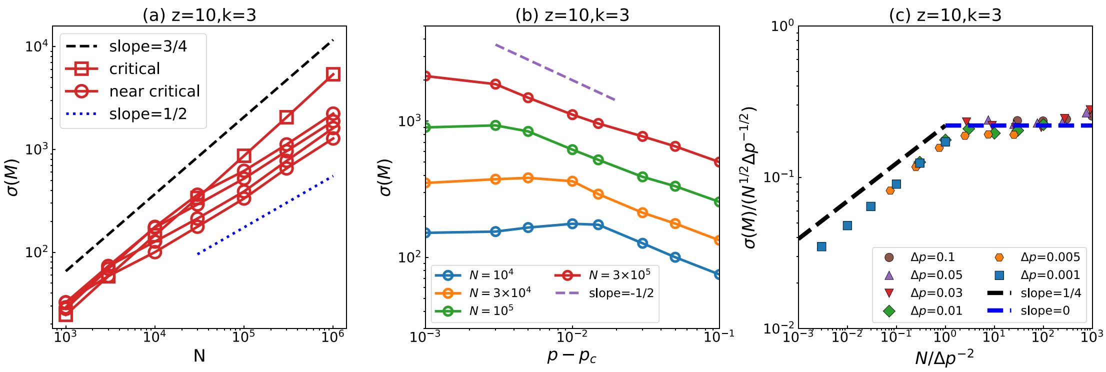

Figure B2. Critical exponents of fluctuations of the critical mass M_c of k-core percolation in ER network near p_c. (a) Scaling relation between the fluctuations of M and the network size N at and near the critical threshold p_c. Here we choose $\Delta p = p - p_c$ from up to down to be $\{0.01, 0.015, 0.03, 0.05\}$. The crossover in the exponent $\widetilde{d_f}$ is clearly seen. (b) Scaling relation between the fluctuations of M and $p - p_c$. (c) Fractal fluctuation behavior in the standard deviation of the mass M near the critical mixed-order percolation threshold, supporting equation (16). It follows the same scaling function, crossover and critical exponents as found in interdependent networks [36].

B.3. Scaling behaviors of mean plateau time in k-core percolation

For a given N we study in figure B3(a) the NOI which we call the mean plateau time $\langle\tau\rangle$ at and near the threshold p_c for different realizations, and then plot it in figure B3(b) as a function of N. The results suggest the following scaling with N,

$$\langle\tau_c\rangle \propto N^{1/3}, \tag{B.1}$$

in agreement with Lee *et al* [18]. While equation (B.1) is valid at the critical p_c, away from the critical regime the mean plateau time is observed in figure B3(b) independent of N, i.e. as $\sim N^0$. Close to the critical threshold, a crossover between these two behaviors is observed and can be described via a scaling function $f_1(\mu_1)$ as

$$\langle\tau\rangle \propto N^{1/3} f_1(\mu_1), \tag{B.2}$$

with

$$\mu_1 = (p_c - p)^{\alpha_1} \cdot N. \tag{B.3}$$

Here, $f_1(\mu_1)$ is a piecewise function satisfying $f_1(\mu_1) \propto$ constant for $\mu_1 < 1$ and $f_1(\mu_1) \propto \mu^{m_1} = (p_c - p)^{\alpha_1 m_1} \cdot N^{m_1}$ for $\mu_1 > 1$. Thus, we have:

$$\langle\tau\rangle \propto \begin{cases} N^{1/3} & \text{for } \mu_1 < 1 \\ N^{1/3+m_1}(p_c - p)^{\alpha_1 m_1} & \text{for } \mu_1 > 1. \end{cases} \tag{B.4}$$

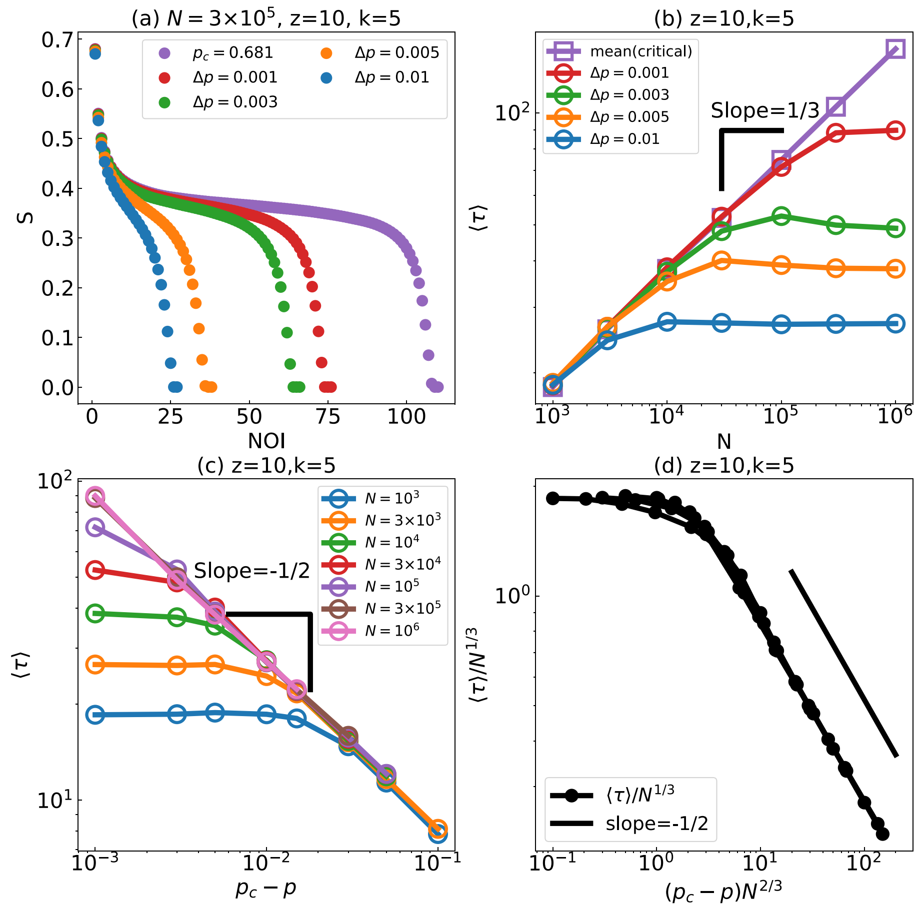

Figure B3. The plateau of single realizations. (a) Dynamical process, i.e. the plateau behavior of the fraction of k-core giant component, S, of a network with mean degree z and size N with NOI (time) of k-core giant component at criticality, p_c and just below p_c. (b) Scaling relations between the mean plateau time (number of iterations-τ) and the network size N at critical p_c and at four non-critical p values just below p_c. Note that, the relation $\langle\tau_c\rangle \propto N^{1/3}$ was also found in [18], supporting the findings here. (c) Scaling relations of the mean plateau time, $\langle\tau\rangle$, against $p_c - p$ for different N. (d) The scaling collapse of (b) and (c), supports equation (B.5). It is seen that the scaling behavior of the mean plateau time and their critical exponents for k-core are identical to those of interdependent networks [35].

From figure B3(b) we can see that $\langle\tau_c\rangle \propto$ constant for $\mu > 1$, implying $1/3 + m_1 = 0$, i.e. $m_1 = -1/3$. Thus, we have:

$$\langle\tau\rangle \propto \begin{cases} N^{1/3} & \text{for}\, \mu_1 < 1 \\ (p_c - p)^{-\alpha_1/3} & \text{for}\, \mu_1 > 1. \end{cases} \tag{B.5}$$

In order to determine the value of α_1, we plot $\langle\tau_c\rangle$ against $p_c - p$ in figure B3(c). We can see that $\langle\tau_c\rangle \propto (p_c - p)^{-1/2}$, implying $\alpha_1 = 3/2$. Finally, to support equation (B.2) and the obtained value of α_1, we created a scaled plot shown in figure 4(d), depicting $\langle\tau_c\rangle / N^{1/3}$ against $N^{2/3}\Delta p$. As can be seen, one achieves a satisfactory scaling collapse with $\alpha_1 = 3/2$, i.e. we have:

$$\langle\tau\rangle \propto \begin{cases} N^{1/3} & \text{for}\, N < N_1' \propto (p_c - p)^{-3/2} \\ (p_c - p)^{-1/2} & \text{for}\, N > N_1'. \end{cases}, \tag{B.6}$$

showing further that the scaling behavior of the mean plateau time and its critical exponents in k-core are identical to those of IN [35].

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