Interdependent Spatially Embedded Networks: Dynamics at Percolation Threshold

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***Abstract*—Spatially embedded systems of interdependent networks with full dependency ($q = 1$) have been found to have a first-order percolation transition if the dependency link length (the maximum distance in lattice units between a node in one network and the node that it depends on in another network) is longer than a certain critical length $r_c \approx 8$. We find here that a similar result is valid for any finite value of q with a larger r_c as q decreases. We also provide a theoretical approach which correctly predicts the relationship between r_c and q. We also examine the dynamics at the percolation threshold p_c for varying r and q and find that there are three different mechanisms of failure for every q depending on r. Below r_c the system undergoes a continuous transition similar to standard percolation. Above r_c there are two distinct first-order transitions for finite or infinite r, respectively. The transition for finite r is characterized by spreading of node failures through the system while the infinite r corresponds to a non-spatial cascading failure similar to the case of random networks. These results extend previous results on spatially embedded interdependent networks to the more realistic cases of partial dependency and shed new light on the specific dynamics of cascading failures in such systems.**

The structure and function of complex networks have been studied extensively over the last fifteen years[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. The central subject of this body of work has been the single network, ignoring possible relationships between networks. In the real world, networks rarely appear in isolation and the relationships between networks can cause new and unexpected phenomena.

Recently, the focus of network research has expanded to interdependent networks characterized by connectivity and dependency links [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24].

The introduction of interdependence has been shown to change the percolation transition from second-order to first order in numerous scenarios [25], [26], [27]. Most of this research, however, has focused on random and scale-free networks without spatial restrictions even though many networks of interest—including all infrastructure networks—are embedded in space. For example, the power grid and the communications network are both embedded in the same two dimensional space and they rely on one another to function: the communications stations require electricity and the power stations require control from the communications network. This interdependency introduces the possibility for cascading failures in which system damage that would otherwise be limited in scope spreads to the entire system causing total collapse.

Understanding the conditions that lead to percolation transitions in interdependent embedded networks is crucial to designing more robust infrastructures. Recently, the case of spatially embedded networks has been studied but only in the case of full dependency ($q = 1$)[28]. The parameter q represents the fraction of nodes in one network that depend on nodes in the other network. In that work, it was shown that there is a critical value of the dependency length $r_c \approx 8$ above which first order transitions appear. For interdependent random networks, it has been shown that there is a finite critical value q_c below which first order transitions are not possible [25]. It has also been shown [29] that $q_c = 0$ when there is no limitation on the dependency link length r or in other words $r = \infty$. In this work we show that for every finite q there exists a finite length r_c such that if $r \geq r_c$ a first-order transition appears. We find that r_c increases as q decreases but for every $q > 0$ there exists a first-order transition if $r \geq r_c$. When q is reduced to zero, standard lattice percolation is recovered. We develop a theory based on [28] which enables accurate prediction of the value of r_c for any value of q.

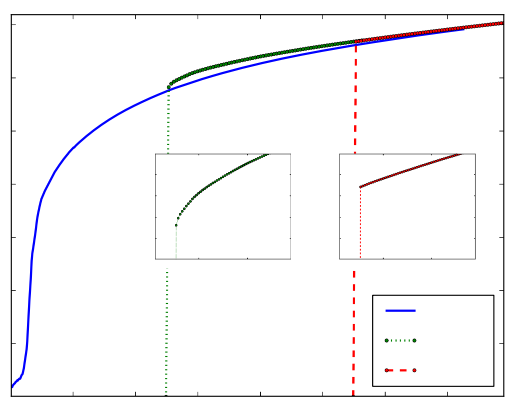

Fig. 1. As r increases, the system collapse changes qualitatively from a second order transition for low r to an abrupt first order transition above r_c. For large but finite values of r, the slope of $P_\infty(p)$ is linear near and above p_c whereas for $r = \infty$, $P_\infty(p)$ shows scaling behavior. Results shown are for $q = 0.7$ with lattice linear size $L = 2900$.

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I. System Description

Our model of two interdependent spatially embedded networks is realized via a pair of square ($N = L \times L$) lattices A and B with periodic boundary conditions. A fraction q of randomly chosen nodes in each network are made dependent on nodes in the other network. Dependency is taken to mean that if a node in network A is removed from the system and a node in B depends on it, that node will be removed as well. All of the dependency links in our model are mutual. This is known as the "no-feedback condition" and prevents single failures from initiating a string of failures that could destroy the entire network[30]. The spatiality is determined by the lattice structure of the networks and by the parameter r with the condition that the length of the dependency link be less than r. Thus for nodes $i \in A$ and $j \in B$ with lattice coordinates (x_i, y_i) and (x_j, y_j) to be interdependent their respective coordinates must satisfy $|x_i - x_j| \leq r$ and $|y_i - y_j| \leq r$. When $r = 0$ we return to the case of standard percolation on a single lattice and when $r = \infty$ we have a mixed case where the networks are spatial but the dependency links are random, as described in [29].

The percolation transition is studied by removing a random fraction $1-p_0$ of nodes (along with the links attached to them) from both networks simultaneously. Then, on each network, clusters which are detached from the largest connected component are removed. After that, the nodes in each network which have lost their supporting node in the opposite network are removed. This in turn causes more clusters to break off from the giant component and the process is continued until no more clusters break away. We use the term *cascade lifetime* or *number of iterations* (NOI) to refer to the number of iterations that this process undergoes until no more nodes fail.

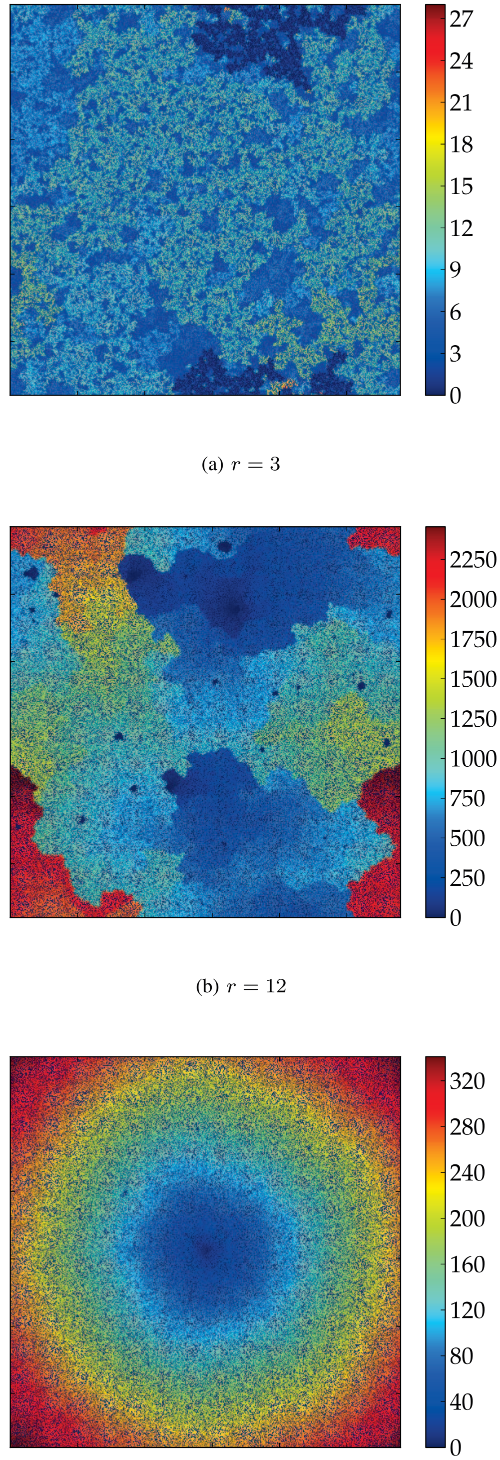
II. Simulation Results

Upon bringing the system to the percolation threshold, we found three distinct mechanisms by which the giant component collapses. For all q, they vary according to r, see Figures 1 and 2.

For $0 < r < r_c$, the the giant component decreases continuously and the percolation transition is similar to standard lattice percolation. However the value of p_c increases monotonically with r (see Fig. 3), reflecting the greater susceptibility to collapse due to the feedback caused by the dependency links.

Once $r \geq r_c$ but still finite, the system undergoes an abrupt first order transition. We determine a realization to undergo a first-order transition if the removal of a single node causes 40% or more of the system to collapse. The transition is not only discontinuous at p_c, it lacks any critical behavior above p_c, (Figure 1). This is due to the spreading process which drives the cascading failure and will be explained below. As r increases more, p_c decreases and asymptotically approaches its value at $r = \infty$.

At $r = \infty$ the value of p_c is similar to large but finite r values but the critical behavior is very different. This is visible in the scaling of $P_\infty(p)$ above p_c, (Figure 1) as well as the



(a) $r = 3$

(b) $r = 12$

(c) $r = 20$

Fig. 2. The nodes of the network colored according to the iteration number in which they failed, at criticality. The $r = 3$ case is similar to percolation in a single lattice, where the giant component near criticality is a fractal [31]. The $r = 12$ case demonstrates the transitional failure caused by partial spreading. Note in particular the extremely long time-scale for $r = 12$ relative to the other cases, cf. Figure 6. The $r = 20$ case shows the highly regular cascading failure driven by a stable spreading process. The realizations shown here have $q = 0.7$, $L = 2900$ and $r_c = 13$.

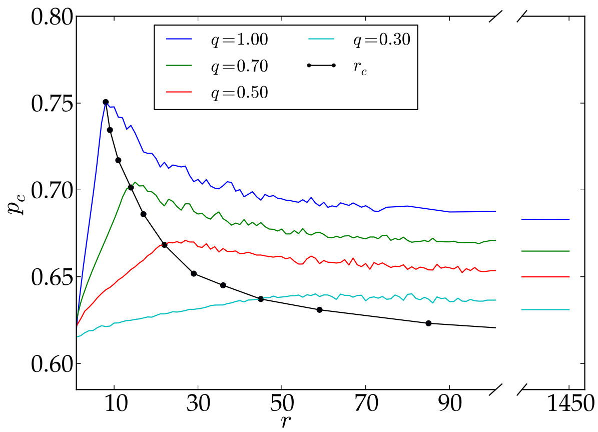


Fig. 3. The percolation threshold p_c, as a function of r for several values of q. As q decreases, the maximal value of p_c decreases and the minimal dependency length for first order transitions (r_c) increases. The simulations were determined to be first order if, on average, the removal of a single node caused more than 40% of the system to collapse. ($L = 2900$)

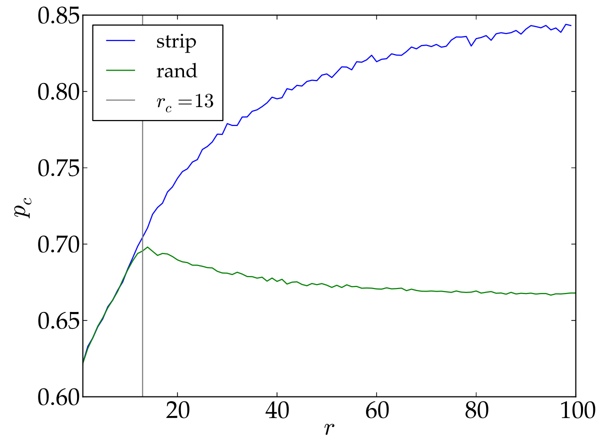


Fig. 4. Comparison between p_c values with and without a strip. For $r < r_c (= 13)$, the strip cannot propogate and its effect on the system behavior as a whole is negligible. When $r > r_c$ but no initial strip is removed, the system is susceptible to propogation but remains in a metastable state until p is low enough for a sufficiently large hole to appear due to random fluctuations. Detailed analysis of the metastable region for the case of $q = 1$ can be found in [32]. The r_c line is determined by the theoretical calculation above, (Eq. 6). These results have $q = 0.7$ and $L = 1000$.

dynamics of the cascading failure which are described below in Figures 7 and 8.

These three regimes are discernible for all values of q which we simulated (0.1 to 1 in steps of 0.1). However, as q decreases the maximal vulnerability ($\max_r p_c$) decreases and r_c increases as is evident in Figure 3. To understand why this happens, we need to understand the type of transition which is taking place and the conditions necessary for it to occur.

The area between the two curves in Fig. 4 is metastable. If a strip of width r were to be removed from one of the embedded networks it would propagate and destroy the entire system. However, the spontaneously generated holes are not large enough to trigger such a cascade. In [32], this region was studied in detail for the case of full dependency and localized attacks of characteristic size r_h.

III. Conditions Necessary for a First-Order Transition to Occur

The first order transition described above is caused by a process of damage spreading. A hole in the lattice emerges from random fluctuations and grows larger from iteration to iteration until it destroys the whole system. We find that this process only occurs if $r \geq r_c$ and that the value of r_c increases as q decreases and diverges at $q = 0$, see Figures 3 and 5.

Because the spreading phase of the cascading failure is triggered by the emergence of a large hole, the conditions necessary for its propagation are essentially the same as those for the propagation of a flat interface. We anticipate that the value of r_c for which a strip can propagate is the same for spontaneous propagation of holes. This assumption is confirmed numerically in Figs. 4 and 5.

A flat interface will propagate through the system if the nodes bordering it have a local $p < p_c$ in a region large enough for clusters to break away at each iteration. To calculate when that will occur, we examine the case of a strip of width r removed from a network. We choose a width r because wider strips will not cause more damage given that the maximal length of a dependency link is r. Increasing the size of the strip to wider than r would have no impact on the dynamics and would only make the system smaller.

Between the interface and the maximal extent of its influence (r), there is a gradient of p values. Depending on the system p and the strength of interdependence q, a fraction of the gradient can have local $p < p_c$. In order for the strip to propogate, it is not sufficient to have a region with $p < p_c$. There needs to be enough space in the $p < p_c$ segment to allow smaller clusters to develop and separate.

If there is insufficient space in the critical region for smaller clusters to separate, the rest of the lattice remains intact and the size of the strip remains constant. However, if there is enough space, the strip will grow from step to step until it overwhelms the system. This theoretical consideration permits the calculation of a dependency length r_c below which there will only be second-order transitions and above which first-order transitions will occur.

In [28] an attack on one network with $q = 1$ was studied. Here we study an attack caused by the removal of $1 - p_0$ nodes from both networks because the results converge faster for $q < 1$ than they do for the single network attack. Both methods are equivalent and differ only in the value of the effective p. To compare results, we must discuss the effective attack strength which can be calculated as the probability that a node is alive given a random attack p_0 on both networks:

$$p = p_0(1 - q + qp_0). \tag{1}$$

To calculate the value of r_c for a given q we begin by evaluating the probability of survival at a distance ρ from the interface ($x_i = \rho + r$). Assuming a random fraction $1 - p_0$ of nodes are removed along with all nodes with $x_i < r$ (from both networks):

$$p(\text{alive}|q, r, \rho, p_0, G) = p_0 \left(1 - q + qp_0 \left[1 - G(\rho, r)\right]\right), \quad (2)$$

in which G denotes the probability that a node at distance ρ in a system with dependence length r has its dependency linked neighbor in the removed region and as such it is defined by the geometry of the attack. In the case of a strip attack (of maximal width r), we obtain $G(\rho, r) = \frac{r-\rho}{2r+1}$, the $+1$ terms owing to the discreteness of the interlink calculation. $G(\rho, r)$ is defined for $0 \geq \rho \geq r$ and must be taken as identically zero for $\rho > r$.

$$p(\text{alive}|q, r, \rho, p_0, \text{strip}) = p_0 \left(1 - q + qp_0 \frac{r + \rho + 1}{2r + 1}\right). \quad (3)$$

Using this, we can calculate ρ_c, the distance from the strip at which the lattice occupation probability is equal to p_c:

$$\rho_c(q, r, p_0) = \frac{2r + 1}{qp_0^2} \left((q - 1)p_0 + p_c\right) - r - 1. \quad (4)$$

Ignoring the discrete correction ($2r + 1 \to 2r$) and taking the $q = 1$ limit, we recover the result from [28] (with p^2 in place of p following (1)). Depending on q and p_0, equation 4 may have no solution. This corresponds to the situation in which even the nodes bordering the strip have $p > p_c$.

If equation (4) has a finite solution, propagation is still not assured because there needs to be enough space for small clusters to fully develop and break away. The smaller clusters that are formed in the $\rho < \rho_c$ ($p < p_c$) region will have a typical size of $\xi_<(\bar{p})$ in which $\bar{p}(q, r, p_0) = \frac{1}{\rho_c} \int_0^{\rho_c} p(\text{alive}|\rho) d\rho$ is the average value of p in the region between the interface and ρ_c and $\xi_<(p)$ is the average correlation length in a pure lattice below p_c. Following [33], [31], we calculate $\xi_<(p)$ as

$$\xi_<^2 = \frac{1}{N_p} \sum_{(i,j)} |\mathbf{r}_i - \mathbf{r}_j|^2, \quad (5)$$

where (i, j) refers to nodes i and j which are in the same connected component and N_p is the total number of such pairs of nodes.

In order for the clusters to break away from the giant component we require that:

$$\rho_c > \xi_<(\bar{p}). \quad (6)$$

The precise structure of the clusters is a gradient percolation problem [34] but a mean field approximation affords sufficient accuracy to determine r_c. In the case of a strip attack, $p(\text{alive})$ is linear in ρ and so we can state:

$$\bar{p} = p(\text{alive}|\rho_c/2). \quad (7)$$

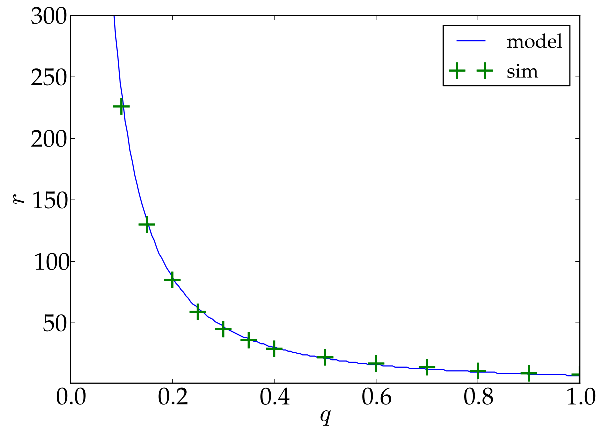


Fig. 5. For every value of q there is a critical dependency length r_c such that for $r > r_c$ the system has a first order transition. The simulations were determined to be first order if, on average, the removal of a single node caused more than 40% of the system to collapse ($L = 2900$).

Since both sides of equation (6) are dependent on q, r and p_0, for every value of q and r we can solve for p_0. If the value of p_0 is low enough to cause the effective p (equation 1) to be below p_c, then strip propagation will not take place because the system will be destroyed via conventional percolation before the conditions necessary for propagation appear. For any q, as we increase r from the standard percolation case at $r = 0$, there is no solution to (6) until a given value which we call r_c. That value increases as the q values under consideration decrease.

To understand this process qualitatively, consider what happens when we decrease q. The minimal p value in the affected region (obtained at $\rho = 0$) increases and since the maximal value must be above p_c, the steepness of the gradient decreases. With a shallow gradient, the whole system needs to be very close to p_c in order to obtain a finite ρ_c. However, since $\xi_<(p)$ diverges near p_c, the requirement that clusters be small enough to break off becomes impossible to fulfill and the system undergoes a typical second-order percolation transition. A first-order transition is made possible by either increasing q (steepening the gradient so that $\bar{p}$ decreases) or increasing r (making it easier for large $\xi_<$ values to fit in the $\rho < \rho_c$ region). Thus for every q there is an r_c above which propogation of a strip is possible, leading to an abrupt first-order transition.

As is visible in Figure 4, the calculated r_c matches the point where p_c for a strip and a random attack begin to diverge. For $r < r_c$, the critical behavior of the system is unaffected by the removal of the strip. The gap between the two curves in Figure 4 shows the introduction of a new kind of cascading failure via spreading. Furthermore, it demonstrates the extreme vulnerability of interdependent spatially embedded networks. For instance, if a strip occupying $\approx 4\%$ of the total system is removed from a system with $p = 0.8$, $q = 0.7$, $L = 1000$

and $r \approx 40$, it will spread and destroy the entire system.

Since the conditions for propogation of a strip and a large hole are almost identical, we expect the result to predict first order transitions in systems under random attack as well. Indeed, we find excellent agreement between our predicted r_c and the measurements obtained from simulated random attacks, (Figure 5).

In Figure 3, the value of p_c decreases as r increases in the $r > r_c$ region. The reason for this decrease is that the hole size necessary to trigger a cascade increases with r. Consider two holes of the same size in systems with different r values. The nodes surrounding the hole will be less affected by the hole in the system with higher r because the probability that their interdependent neighbor is in the affected area is lower. When the focus is restricted to random attacks, this indicates that the system robustness increases with r. However, if we consider the effects of a targeted attack like the strip attack in Figure 4, we see that the robustness decreases with r.

Another important value which we can now calculate is $\tilde{q}$, the degree of interdependence below which a strip cannot propagate as long as the rest of the system is fully occupied ($p_0 = 1$). To find this value, we take $p_0 = 1$ and $\rho_c = 0$ in (4). Letting $r \to \infty$, we can obtain $\tilde{q}$

$$\tilde{q} = 2(1 - p_c) \approx 0.8146 \tag{8}$$

For $q < \tilde{q}$, a strip will never propagate as long as the system $p = 1$. For $q > \tilde{q}$, a strip can propagate even with $p_0 = 1$, if r is large enough.

IV. Dynamics of Cascading Failures at p_c

Previous work was limited to the case where $q = 1$ and $r_c \approx 8$. As such the range of r for the second-order regime ($r < r_c$) was limited and some of the dynamic properties were difficult to establish, particularly with regard to the transitional state of partial spreading which we observe as r approaches r_c. Taking $q < 1$, the range of r values below r_c increases and the dynamic properties at criticality are more visible.

To understand the different regions described above, it is necessary to examine the dynamics at p_c for each regime. Though all of the percolation-like transitions for $r > 0$ consist of cascading failures, the specific dynamics vary significantly for different values of r. For $r < r_c$, the number of iterations at p_c is greater than zero but remains low, scaling weakly with system size. the system falls in large clumps that get cut off from the giant component. Though the nonzero cascade lifetime differentiates this process from standard percolation, it is qualitatively very similar as is evident in Figures 1 and 2a. Below r_c, the conditions for the spreading of a large hole are not met, as explained in the previous section. There is insufficient space near the interface for small clusters to break off. However, due to random fluctuations in the system, there is a nonzero probability that some clusters will break off. Because the conditions to guarantee the spreading are not met, a single large hole is not enough to destroy the system. Instead, multiple interfaces are formed and spread irregularly. This leads to an irregular propagation pattern characterized by periods of propagation followed by long periods of inactivity and an extremely long cascade lifetime (Figure 2b). Because the dynamics of the cascading failure are highly sensitive to random fluctuations in dependency link structure and failed node geometry the cascade lifetime (NOI) shows high variability in this region (Figure 6).

Crucially, below r_c a large hole can develop but not propogate. Once $r \geq r_c$, the conditions for the spreading of a flat interface are met and as such the spreading becomes regular and the collapse is triggered by a single hole, see Figure 2c. Once a hole of critical size appears in the system, it

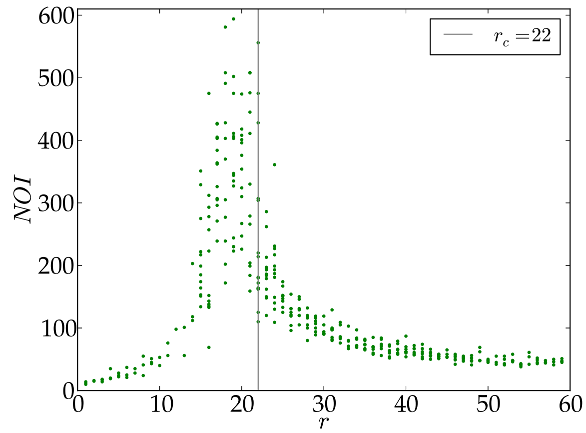


Fig. 6. Scatter plot of number of iterations (NOI) at p_c. As r approaches r_c, the variability of the number of iterations at criticality rises dramatically. This reflects the "stop and start" spreading which characterizes the failure before r is large enough to permit steady propogation. (Here $q = 0.5$ and $L = 1000$)

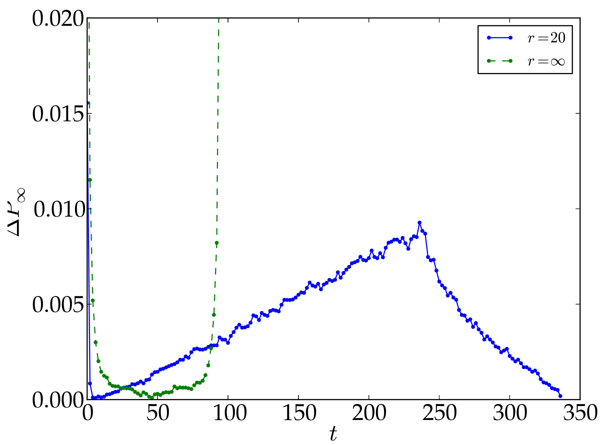


Fig. 7. Fraction of nodes removed per iteration. Though the value of p_c quickly approaches its asymptotic value (Figure 3), the dynamics of the first-order transition are markedly different for large finite r vs infinite. Only with finite $r > r_c$ do we see the signature linear region which corresponds to a spreading hole. ($q = 0.7$, $L = 2900$)

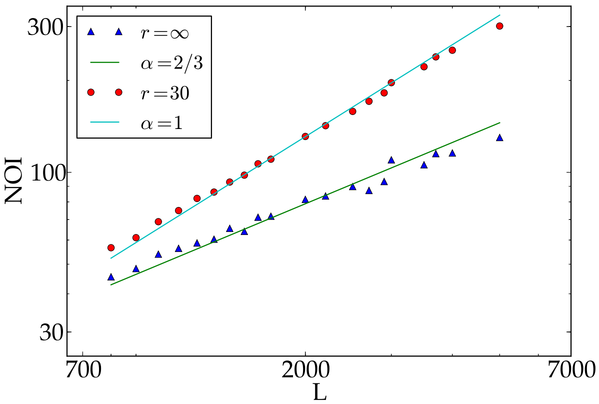


Fig. 8. Scaling of NOI at p_c with system size for large and infinite dependency lengths —$NOI \sim L^{\alpha}$. Finite and infinite r are clearly differentiated when the NOI at criticality is plotted against system size. Finite r shows an approximately linear relation while infinite r is clearly sublinear.

will spread uniformly in all directions until the entire system collapses. This regularity decreases the length and variability of the cascade lifetime, see Figures 6 and 2c. Because the spreading requires only one hole of critical size, it can be triggered by the removal of a single node and is thus first-order. This dynamics holds even for $r \gg r_c$.

When $r = \infty$, i.e. the size of the system, the dynamics of collapse are once again different. The cascading failure in this case is driven not by clusters breaking away but by the expectation of failing nodes per node removed, as detailed in [35]. When this expectation ≈ 1, a chain of failures will spontaneously be initiated. The failures remain very small but as the overall system p decreases the rate of failing nodes per node removed increases. Once that value is stably above 1, it begins to grow exponentially and the entire system collapses within a few steps. In [35] it was demonstrated that cascades driven by this process scale as $NOI \sim N^{1/3} = L^{2/3}$, markedly different from the linear ($NOI \sim L$) behavior expected in the uniformly expanding circle. Indeed we see the difference between the finite r linear spreading cascade and the plateau-like $r = \infty$ cascade for individual realizations in Figure 7 as well as the expected difference in scaling with respect to system size in Figure 8.

V. Discussion

Dependencies between embedded networks are ubiquitous but not necessarily total. Certain nodes may not be dependent (consider a communications station with on-site power generation) and it is crucial to understand to what extent the risk for catastrophic cascades is mitigated by decreased dependency. Furthermore, $q < 1$ is equivalent to the scenario in which the dependency is not deterministic but rather the failure of a node in one network causes an increased probability of failure for a node in another network. From a theoretical point of view, understanding critical behavior for $q < 1$ is an important preliminary result for extending spatially embedded networks to networks with different numbers of nodes.

We find that the risk does decrease but it in no way disappears. The susceptibility to a spreading catastrophic cascade is tied to the distance between interdependent nodes (in lattice units). The spreading cascade is doubly dangerous due to its extreme suddenness. As p is decreased to p_c, there is no indication that a cascade will begin until it has destroyed the entire system (Figure 1, right inset).

Interdependent networks can avoid spreading catastrophic cascades only if the distance between interdependent nodes is below r_c, though increasing the distance to a value much greater than r_c can also provide a small increase in robustness (Figure 3) with respect to random attacks.

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