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Optimization of robustness based on reinforced nodes in a modular network

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Abstract – Many networks such as critical infrastructures exhibit a modular structure. One approach to increase the robustness of these systems is to reinforce a fraction of the nodes so that the reinforced nodes provide additional needed sources for themselves as well as for their nearby neighborhood. Since reinforcing a node can be expensive, the efficiency of the decentralization process by reinforced nodes is vital. Here we develop a model which combines both modularity and reinforced nodes and study the robustness of the system. Using tools from percolation theory, we derive an analytical solution for the robustness resulting from any partition of reinforced nodes; between nodes that have links that connect between modules and nodes which have links only within modules. We find that near the critical percolation threshold the robustness is greatly affected by the partition. In particular, we find a partition of reinforced nodes that yields optimal robustness and we show that the optimal partition remains constant for high average degrees.



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Introduction. – In recent years, much attention has been focused on the resilience and stability of networks having a community structure [1–5]. Examples of real-world networks with a community structure are the brain [6–8], infrastructures [9,10] and social networks [11–13] as well as many others [14–18].

Here we focus on a recently proposed community model [19] where only a fraction r of nodes are capable of having inter-links that connect them to other modules (communities). This is a realistic model since in many real modular networks, such as power poles in different cities, some of the power poles connect only to poles from their city and only specific power poles have in addition connections to poles from other cities. The resilience of a network can be estimated by the giant component (GC) size. In percolation theory [20–26], the GC is defined as the network's largest connected component and its size is considered as the system's order parameter. The GC represents the network's resilience since when nodes are not connected to the GC they are not regarded as functioning, as they cannot communicate or get resources from other nodes. This condition that functioning depends on being connected to the GC can be regarded as a centralization feature.

Recently, a new concept of reinforced nodes has been introduced into modeling of real-world networks [27]. The

reinforced nodes are nodes that have their own support and can also support the cluster of nodes connected to them. For example, in the internet network, communication satellites [28] or high-altitude platforms [29] can serve as reinforced nodes and support important internet ports in cases of connection failures. Thus, the concept of reinforced nodes can be regarded as a decentralization feature, since there are nodes that are not in the GC but can still function properly. Considering reinforced nodes, the new order parameter which expresses the functionality of the system can be taken as the *functional* component and not the known *giant* component [27]. The functional component (FC) contains both the giant component and smaller components which include at least one reinforced node, see fig. 1 and the stability of the system is characterized by the size of the FC. Interestingly, it has been found [27] that in a regular, non-modular network a very small fraction of reinforced nodes increases the robustness significantly.

Here, we study the stability of modular systems in the presence of reinforced nodes. In particular, we distinguish between reinforced nodes that have links to other modules (inter-connected nodes) and reinforced nodes that have links only to their module (intra-connected nodes). We find, using both theory and simulations, the functional component size, and we address the following optimization question: where to place the reinforced nodes in order to optimize the robustness of the system to random failures

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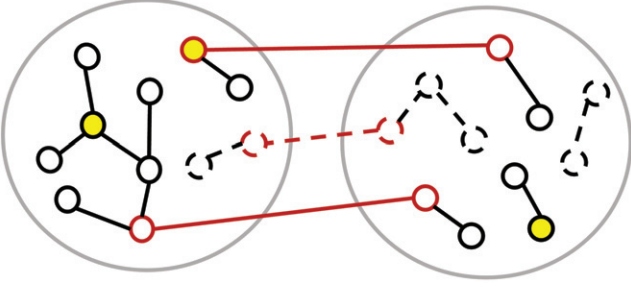


Fig. 1: A schematic representation of the model with $m = 2$ modules. Each grey circle represents a module. The black and red lines represent the intra- and inter-links, respectively. The small circles are the nodes, where the inter-connected nodes are colored in red and the reinforced nodes are marked in yellow. The functional component contains the giant cluster and the clusters which contain at least one reinforced (yellow) node. Nodes and links which are not part of the functional component are marked by dashed lines.

(*i.e.*, to obtain the largest FC)? Specifically, the question is how to distribute optimally the reinforced nodes between intra-connected nodes and inter-connected nodes. An example of a system that motivates our research is the power grid network which is composed of electric generators. In this example, cities can be regarded as communities and the reinforced nodes are the electric generators. Thus, the optimization question is how to distribute the electric generators between the inter-connected power poles that connect different cities and the intra-connected power poles that do not.

Model and theoretical approach. – Our network model, of size N , consists of m ER modules (communities) where each module has N/m nodes. The links within the modules are called “intra-links” and the links which connect nodes from different modules are called “inter-links”. In all modules, the average degree considering only the intra-links is z . The nodes which can have inter-links are called “inter-connected nodes” and we select them randomly with probability r . For each pair of modules, A and B for instance, each inter-link is randomly placed between an inter-connected node from A and an inter-connected node from B . Thus, for placing M_{inter} links between any two modules, the degree distribution of the inter-connected nodes —when considering only their inter-links— is Poissonian with average inter-degree $\kappa = mM_{inter}/rN$. Our network model includes also a fraction ρ of reinforced nodes.

Next, we derive analytically the size of the FC as a function of the number of reinforced nodes, first for a random distribution of reinforced nodes and later for a particular distribution of them. We use tools of percolation theory and define the generating functions considering the intra- and inter-links as follows: $G_0^{intra}(x) = \sum_k p_k^{intra} x^k$ and $G_0^{inter}(x) = \sum_k p_k^{inter} x^k$ for the degree distribution and $G_1^{intra}(x) = \sum_k q_k^{intra} x^k$ and $G_1^{inter}(x) = \sum_k q_k^{inter} x^k$ for

the excess degree distribution. In these functions, p_k^{intra} is the probability for a node to have k intra-links and q_k^{intra} is the probability for a node, reached by following a link, to have k intra-links. p_k^{inter} and q_k^{inter} are the same as p_k^{intra} and q_k^{intra} only for inter-links [30]. We define u as the probability that an intra-link leads to a node that is not part of the FC; v as the probability that an inter-link leads to a node that is not part of the FC, and S as the probability that a node is part of the FC. Thus, for a network with a fraction of reinforced nodes, ρ , which is distributed randomly, the probabilities u , v and S satisfy the equations

$$\begin{aligned} 1-u &= p \left[1-(1-\rho)G_1^{intra}(u) \left[1-r+r \prod_{i=1}^{m-1} G_0^{inter}(v) \right] \right], \\ 1-v &= p \left[1-(1-\rho)G_0^{intra}(u) \prod_{i=1}^{m-1} G_1^{inter}(v) \right], \\ S &= p \left[1-(1-\rho)G_0^{intra}(u) \left[1-r+r \prod_{i=1}^{m-1} G_0^{inter}(v) \right] \right], \end{aligned} \quad (1)$$

for the case of randomly removing a fraction $1-p$ of the nodes from the network.

In our model, the inter-links distributions are Poissonian with average inter-degree κ , therefore $G_0^{inter}(x) = G_1^{inter}(x) = e^{-\kappa(1-x)}$. In addition, in the limit of infinitely large ER networks, the degree distributions for the intra-links are Poissonian with an average degree z and thus $G_0^{intra}(x) = G_1^{intra}(x) = e^{-z(1-x)}$. These equations (for the random case) lead to a single transcendental equation relating S , r and ρ ,

$$\begin{aligned} e^{-zS}(r-1)(1-\rho)+1-\frac{S}{p} &= r(1-\rho)e^{-zS} \\ \cdot \exp \left[\frac{(m-1)p\kappa \left(e^{-zS}(r-1)(1-\rho)+1-\frac{S}{p}-r \right)}{r} \right]. \end{aligned} \quad (2)$$

Before presenting the main equations of this paper, we note that for $m = 1$, eq. (2) coincides with the equation, derived by Yuan *et al.* [27], for a network that contains only a single ER module. In addition, we note that for $\rho = 0$, eq. (2) coincides with the equation of Dong *et al.* [19] which analyses a modular network without reinforced nodes. Thus, eq. (2) can be regarded as a generalization for percolation of modular networks in the presence of reinforced nodes.

Next, we further generalize the system by splitting ρ to ρ_x and ρ_o where ρ_x is the fraction of reinforced nodes in the network that are inter-connected nodes and ρ_o is the fraction of reinforced nodes in the network that are intra-connected nodes (*i.e.*, nodes that do not have inter-links), thus, $\rho_x + \rho_o = \rho$. Using these notations we re-derive the equations of u , v and S for any partition of the fraction of reinforced nodes ρ between the intra- and inter-connected

nodes (ρ_o and ρ_x , respectively), as follows:

$$\begin{aligned}
 1 - u &= p \left[1 - G_1^{intra}(u) \left[1 - r - \rho_o + (r - \rho_x) \right. \right. \\
 &\quad \left. \left. \cdot \prod_{v=1}^{m-1} G_0^{inter}(v) \right] \right], \\
 1 - v &= p \left[1 - \left(1 - \frac{\rho_x}{r} \right) G_0^{intra}(u) \prod_{v=1}^{m-1} G_1^{inter}(v) \right], \\
 S &= p \left[1 - G_0^{intra}(u) \left[1 - r - \rho_o + (r - \rho_x) \right. \right. \\
 &\quad \left. \left. \cdot \prod_{v=1}^{m-1} G_0^{inter}(v) \right] \right]. \tag{3}
 \end{aligned}$$

The generating functions for the intra- and inter-links are the ones defined above. Therefore, a single transcendental equation for the functional component relating S , r , ρ_x and ρ_o can be written as

$$\begin{aligned}
 e^{-zS}(r - 1 + \rho_o) + 1 - \frac{S}{p} &= (r - \rho_x)e^{-zS} \\
 \cdot \exp \left[\frac{(m-1)p\kappa \left(e^{-zS}(r - 1 + \rho_o) + 1 - \frac{S}{p} - r \right)}{r} \right]. \tag{4}
 \end{aligned}$$

The solution of eq. (4) will give us the FC, S , for percolation in the presence of ρ_x and ρ_o reinforced nodes.

Results. – For the sake of simplicity, here we analyze a network with $m = 2$ modules, while in the Supplementary Material [Supplementarymaterial.pdf](#) (SM) we present the results for the general case of m modules. First, we quantify the resilience of a network where the reinforced nodes are positioned randomly, by obtaining S for different values of p , r and ρ , both by solving eq. (2) and by numerical simulations. In fig. 2, we present S as a function of p , and show that S increases with both the increase of r and the increase of ρ . As seen, the analytical solution is in a very good agreement with the results obtained from the numerical simulations.

In addition, it was found that non-modular ER networks without reinforced nodes undergo a second-order percolation phase transition at $p_c = 1/z$ [31,32]. Here, it can be seen that in the presence of inter-links and reinforced nodes the percolation phase transition disappears and becomes a transition-free behaviour (fig. 2). This means that r and ρ are analogous to the external field in a spin system [19,23,27,33,34].

Next, we study various networks with different distributions of the reinforced nodes between inter-connected and intra-connected nodes. In fig. 3, we show S at criticality ($p = p_c$), as a function of the fraction of reinforced nodes which are inter-connected nodes, ρ_x/ρ . It can be seen that for a given ρ and an average intra-degree z , S as a function of ρ_x/ρ behaves differently for different values of r . For a very small r ($r = 0.01$), S decreases monotonically;

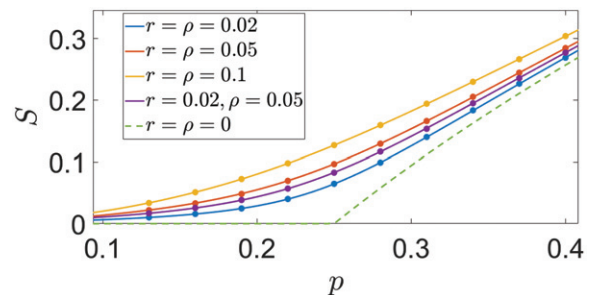


Fig. 2: The size of the FC, S , as a function of p for different r and ρ values where the reinforced nodes are distributed randomly. Lines and symbols denote analytical and simulation results, respectively. The dashed green line represents the theoretical solution of a single ER network without reinforced nodes. For these runs we chose $m = 2$, $N_1 = N_2 = 10^6$, $M_{inter} = N_1$ and $z = 4$.

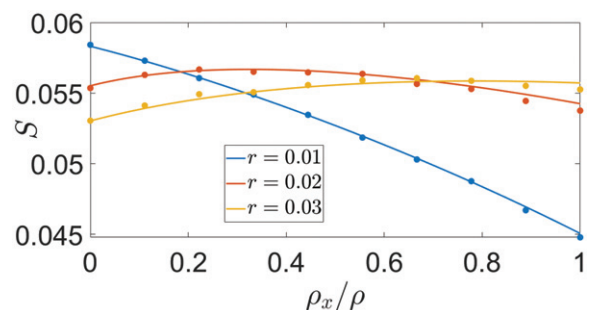


Fig. 3: The size of the FC, S , as a function of ρ_x/ρ , at $p = p_c = 1/z$ (as in ER), for several values of r . Lines and symbols denote analytical and simulation results, respectively. Note that for $r = 0.01$ it is optimal to put the reinforced nodes solely as intra-connected. For $r = 0.03$ solely as inter-connected nodes. However, interestingly for $r = 0.02$ it is optimal to partition the reinforced nodes between the intra- and inter-connected nodes. Here, $m = 2$, $N_1 = N_2 = 10^6$, $M_{inter} = 5 \cdot 10^4$, $z = 3$, $p_c = 0.3333$ and $\rho = 0.01$.

for a slightly larger r ($r = 0.03$), S increases monotonically. However, for intermediate values of r ($r = 0.02$), S behaves non-monotonically, *i.e.*, as a concave function with a maximum. Thus, we conclude that for very small r values the best strategy is to place the reinforced nodes as the intra-connected nodes ($\rho_x = 0$), while for larger r values it is better to place them as the inter-connected nodes ($\rho_x = \rho$). However, as seen in fig. 3, for intermediate r values ($r = 0.02$) the optimal distribution is to share properly the reinforced nodes between inter-connected nodes and intra-connected nodes.

Next, for any given ρ we find its partition of ρ_x and ρ_o which generates the maximal S . We define ρ_x^* as the value of ρ_x which yields the optimal division, *i.e.*, we calculate S , by eq. (4) for different values of ρ_x between 0 to ρ and define ρ_x^* to be the ρ_x value which maintains the maximal S value. We calculate ρ_x^* for different values of p , r and ρ (see fig. 4(a)), while for $\rho \leq r$, ρ_x^* could reach 1. For $p > p_c$, we obtain $\rho_x^* = 0$ (*i.e.*, it is better to reinforce the

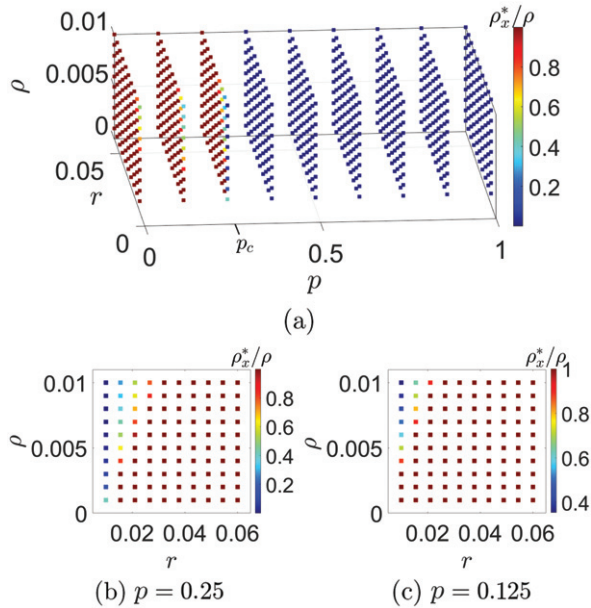


Fig. 4: The value of ρ_x^*/ρ as a function of r and ρ , for $m = 2$, $N_1 = N_2 = 10^6$, $M_{inter} = 5 \cdot 10^4$, $z = 4$. (a) ρ_x^*/ρ for different values of p . For $p > p_c$, the circles are blue which means $\rho_x^* = 0$. (b) ρ_x^*/ρ only for $p = p_c = 1/z$. (c) ρ_x^*/ρ only for $p = 0.5 \cdot p_c$.

intra-nodes) for any values of r and ρ . On the other hand, for $p \leq p_c$, ρ_x^* is determined by r and ρ . For any given ρ , ρ_x^* increases with r , see for instance fig. 4(b) for $p = p_c$ or fig. 4(c) for $p = 0.5 \cdot p_c$. Our results in fig. 4 demonstrate that one can distribute the reinforced nodes between the intra- and inter-nodes such that the robustness is optimal. In the SM we show that the differences in the size of the FC between different divisions of the reinforced nodes are mostly significant for $p \leq p_c$ values. Thus, when demonstrating the optimization question of where to place the reinforced nodes, we focus on the $p \leq p_c$ regime.

In fig. 5, we show that for both $p = p_c$ and $p = 0.9 \cdot p_c$, when we increase the value of the average intra-degree z , ρ_x^* approaches a constant value. Note also that for each value of z we have a different value of p_c , since $p_c = 1/z$. In fig. 5(a) and (b) we set $M_{inter} = 5 \cdot 10^4$ and $M_{inter} = 5 \cdot 10^3$ respectively. It can be seen that for these cases the optimal division of the reinforced nodes is different. For small z values for instance, when there are more inter-links the optimal reinforcement is $\rho_x^* = 0$ (*i.e.*, it is futile to reinforce the inter-connected nodes), while for fewer inter-links, it is optimal to reinforce a fraction of the inter-connected nodes, *i.e.*, $\rho_x^* > 0$. From this we conclude that the less linked the interconnected nodes are, the higher fraction of them should be reinforced.

Summary. – In summary, we have developed a general percolation framework for studying a new realistic network model of m ER modules. We have derived the effect of reinforced nodes on the size of the functional component (FC) of our modular network, *i.e.*, the effect of such a decentralization approach on the network’s robustness.

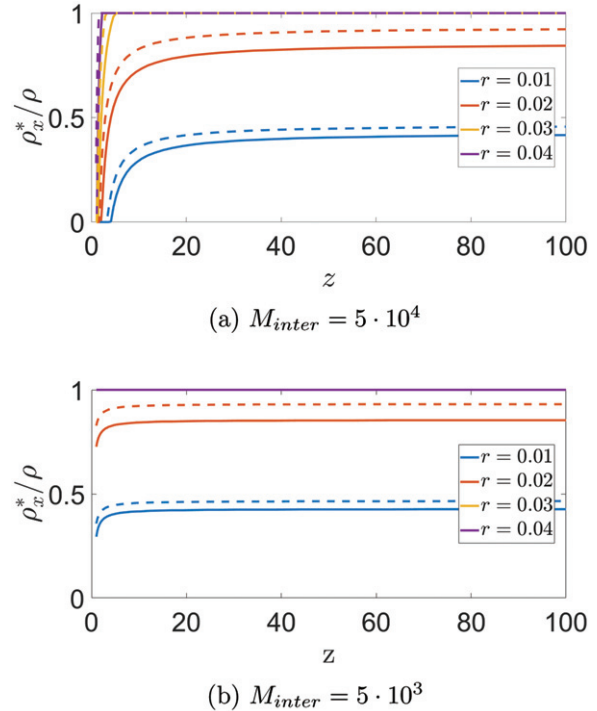


Fig. 5: The fraction ρ_x^*/ρ as a function of z for different values of r . The dashed lines are for $p = 0.9 \cdot p_c$ while the solid lines are for $p = p_c$. Note that the optimal ratio of the reinforced nodes approaches a constant for high intra-degrees. Here, $m = 2$, $N_1 = N_2 = 10^6$, $\rho = 0.01$, (a) $M_{inter} = 5 \cdot 10^4$ and (b) $M_{inter} = 5 \cdot 10^3$.

Previously, the concept of reinforced nodes has been studied only for a non-modular (single community) network and when placing the reinforced nodes at random while here we addressed for the first time an optimization problem of modular networks and non-random locations. We find the fraction of reinforced nodes within the interconnected nodes which provides the largest FC, ρ_x^* , by simulations and theory. We also showed that for a broad range of parameters the value of ρ_x^* is a non-trivial intermediate value (especially near criticality p_c) and becomes constant for high average intra-degrees. These results may have significant practical applications. For example, they can be used to determine the optimal way to distribute the power generators, in a given electricity infrastructure network (which usually has a modular structure). We propose that our novel framework about the effect of reinforced nodes on the robustness of modules connected randomly will be extended in the future to study the robustness of spatial networks of modules, where the modules are embedded in space and only nearby modules are capable of having links between them [35].

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