Critical behavior of cascading failures in overloaded networks

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(Received 25 July 2023; accepted 6 February 2024; published 6 March 2024)

While network abrupt breakdowns due to overloads and cascading failures have been studied extensively, the critical exponents and the universality class of such phase transitions have not been discussed. Here, we study breakdowns triggered by failures of links and overloads in networks with a spatial characteristic link length ζ. Our results indicate that this abrupt transition has features and critical exponents similar to those of interdependent networks, suggesting that both systems are in the same universality class. For weakly embedded systems (i.e., ζ of the order of the system size L) we observe a mixed-order transition, where the order parameter collapses following a long critical plateau. On the other hand, strongly embedded systems (i.e., $\zeta \ll L$) exhibit a pure first-order transition, involving nucleation and the growth of damage. The system's critical behavior in both limits is similar to that observed in interdependent networks.

DOI: 10.1103/PhysRevE.109.034302

I. INTRODUCTION

Cascading failures and system collapse due to overloads have been modeled and studied within a network framework [1,2]. Even a small failure (e.g., deliberate attacks, natural disasters, or random malfunctions) may spread the overloads in relevant infrastructure such as power grids, transportation networks, and communication systems, producing a partial or total collapse. Thus, understanding the laws of cascading failures due to overloads (CFO) is crucial for ensuring the operation of infrastructure and services that we rely on every day. Infrastructure is often embedded in two- or three-dimensional space [3–6] and, far from ideal systems such as lattices, many real-world networks present a characteristic link length ζ [7–9]. Several studies [5,10–13] model this property with a two-dimensional (2D) lattice where the sites are the nodes of the network and link lengths are chosen from an exponential distribution, $P(r) \sim \exp(-r/\zeta)$ (the so-called ζ model), allowing dimension to change from two, for small ζ (short links), to infinite for large ζ (i.e., of order of the system linear size L) [8]. Thus, in the ζ model, the parameter ζ represents the strength of the spatial embedding.

A fundamental model for CFO is the one developed by Motter and Lai (ML) [1] that introduced the concept of load and overload for a node. In this model, load is defined as the number of shortest paths that pass through the node, and is considered a measure of relevance in the transmission of some quantity (e.g., information or energy) throughout the system. They also defined a threshold called capacity, proportional to the initial load and representing the maximum load that a node can hold. Above the capacity, the node becomes overloaded and fails. However, the shortest path is not always the optimal path [14]. A reasonable modification of this model is defining weighted networks, where links have associated weights that may indicate, for instance, the time (or cost) that it takes to traverse a link. In this way, optimal paths, which represent the paths with minimal travel time (or cost) between nodes, are considered to define the loads.

Currently, the critical behavior and the universality class of CFO's phase transition have not been systematically studied. Here, we study this phase transition in both, spatial ζ model [10] and in Erdős-Rényi (ER) [15–17] networks, finding indications that it belongs to the same universality class as percolation of interdependent networks [18–23]. We observe that for weakly or non-spatially-embedded systems, such as ER networks or the ζ model for large ζ (i.e., $\zeta \sim L$), there exists a mixed-order transition, similar to interdependent ER networks [18,19,21,24]. At this abrupt transition, we find a long-term plateau in the order parameter characterized by critical exponents. In contrast, for strongly embedded networks, (i.e., $\zeta \ll L$), we observe a pure first-order transition caused by nucleation of a random damage, a behavior also exhibited by interdependent lattices with finite-length dependencies or spatial multiplex networks [24,25].

II. MODEL

Our system consists of a 2D lattice of size $N = L \times L$ with link lengths r exponentially distributed, i.e., $P(r) \sim \exp(-r/\zeta)$ (ζ model [10]), and average degree $\langle k \rangle$ (self- or multiple connections are not allowed, and we assume periodic

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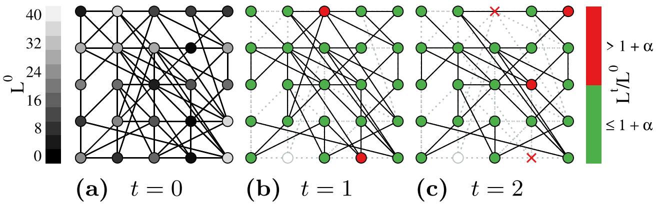

FIG. 1. Model demonstration. (a) A spatially embedded network with $L = 5$, $\langle k \rangle = 4$, and $\zeta = 5$. The load is represented by colors, increasing from dark to light black. (b) Randomly removed links ($1 - p = 0.21$), represented by gray dashed lines, produce changes in the node's loads. For a green node i, $L_i^1/L_i^0 \leqslant 1 + \alpha$, i.e., the load does not exceed the node's capacity, while for a red node j, $L_j^1/L_j^0 > 1 + \alpha$, and the node becomes overloaded and fails. (c) Failed nodes (red crosses) are removed altogether with their links, producing new overloads that continue the cascade process. The tolerance for this network is $\alpha = 2$.

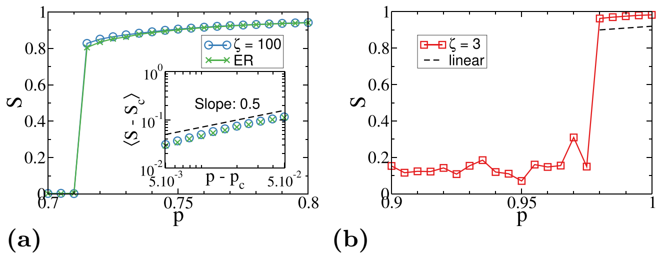

FIG. 2. Giant component of functional nodes, S, as a function of the fraction of nonremoved links, p, for (a) $\zeta = 100$ and (b) $\zeta = 3$. In the inset of (a), we show a dashed line with slope $\beta = 1/2$, which characterizes the system's power-law behavior at criticality for large ζ. The slight deviation from a power-law (with exponent $1/2$) behavior observed when the system is close to criticality (left side of the inset) is due to finite-size effects, and should improve as we increase the linear size of the system, L. Results in the main plots correspond to individual realizations, while the inset plot is obtained by averaging $S - S_c$ over ten runs at each value of $p - p_c$. The remaining network parameters are $L = 350$ (for the ER network $N = 122\,500 = 350^2$), $\langle k \rangle = 4$, and $\alpha = 2$.

boundary conditions). Regarding the CFO dynamics, we study the ML model [1] in weighted networks, with positive weights that follow a Gaussian distribution. We define the load of node i, $L_i(t) \equiv L_i^t$, as the number of optimal paths between all pairs of nodes, excluding i, that pass through i at time t. The maximum load that a node can sustain at any time is given by its capacity, $C_i = L_i^0(1 + \alpha)$, which is proportional to the initial load L_i^0. The parameter α is the system's tolerance, and it represents the resilience of nodes to failure.

At $t = 1$, we randomly remove a fraction $1 - p$ of links, $p \in [0, 1]$. This produces changes of the optimal paths throughout the network, affecting the node's loads, which may generate successive failures due to nodes that become overloaded, in a cascade manner (see Fig. 1). After removing the links, we advance one unit of time and compute the new loads. For $t > 1$, node i fails if $L_i^t > C_i$, we remove failed nodes and their links, and advance one unit of time. We repeat the process until there are no more failures in the network.

The model presented above is not solvable analytically because of spatial constraints, but it can be analyzed via extensive time-consuming numerical simulations. To reduce the sensitivity of the results and produce smoother and consistent curves for a single realization, randomness is somewhat reduced. When performing percolation using a series of $1 - p$ values, we proceed as follows: If E_{p_1} is the set of links that have been randomly removed for $1 - p_1$, then, for a larger value $1 - p_2$, we remove the same set of links E_{p_1} and additional random links until we reach the value $1 - p_2$.

III. RESULTS

We analyze the relative size of the giant component of functional nodes at the end of the cascading process, $S(p) \equiv S$, for weak and strong spatial embedding, i.e., for large and small ζ, respectively. This is shown in Fig. 2. In both cases, we find that the system undergoes an abrupt transition at a critical value p_c, such that $S(p \geqslant p_c) > 0$. Nevertheless, we can distinguish two different behaviors at the vicinity of these transitions. For weak spatial embedding [$\zeta = 100$, Fig. 2(a)] the system approaches criticality, for $p > p_c$ and $S > 0$, with a clear curvature that is absent for strong embedding [$\zeta = 3$, Fig. 2(b)]. We characterize the weakly embedded system through a generalization of the critical exponent β for abrupt transitions [25,26], with respect to $S(p_c) > 0$: $S(p) - S(p_c) \sim (p - p_c)^\beta$, $p \gtrsim p_c$. In the inset of Fig. 2(a), we show that $\beta \cong 0.5$ for $\zeta = 100$, coinciding with the usual mixed-order transition and with interdependent random networks [25]. In contrast, for a strong spatial structure [$\zeta = 3$, Fig. 2(b)], we do not observe a curvature with a critical exponent, but just a linear decrease followed by an abrupt collapse, suggesting a pure first-order transition as in interdependent spatial networks (see, e.g., Fig. 1 in Ref. [22]).

The critical threshold p_c and the mass of the giant component at p_c, $M_c = NS_c$, may vary between realizations (see Fig. 1 of the Supplemental Material [27]). We study their fluctuations, $\sigma(p_c) = (\langle {p_c}^2 \rangle - \langle p_c \rangle^2)^{1/2}$ and $\sigma(M_c) = (\langle {M_c}^2 \rangle - \langle M_c \rangle^2)^{1/2}$, on networks with long-range connectivity links ($\zeta = L$) and different system sizes. Gross *et al.* [6] found for interdependent networks with long-range dependencies that a finite-size scaling analysis yields the relations $\sigma(p_c) \sim L^{-1/\nu'}$, $\nu' = 2/d$, and $\sigma(M_c) \sim L^{d_f'}$, $d_f' = 3d/4$, where d is the spatial dimension. In Fig. 3, we show that for the ML overload model [1] there exists a similar scaling with the linear system's size L, and with the same exponents (i.e., for $d = 2$, $\nu' = 1$, and $d_f' = 3/2$).

Continuing the comparison between spatial and nonspatial networks, we now observe how the CFO evolves while reaching the final state, close to criticality. In Fig. 4(a), we show the time evolution of S for $\zeta = 100$ and several values of p, with $p \leqslant p_c$. The total time of the cascade, τ, increases as the system gets closer to criticality [see also Fig. 5(a)]. These cascades also show a plateau in S, where a microscopic amount of failures [Fig. 4(b)] keeps the cascades going on with a branching factor $\eta \approx 1$ [Fig. 4(c)], for a number of time steps of the order of $N^{1/3}$ [see Fig. 5(b)]. Due to finite-size effects, this phase does not last forever and, eventually, the amount of failed nodes starts to increase because of accumulated damage in the system, leading to an abrupt collapse [20,21].

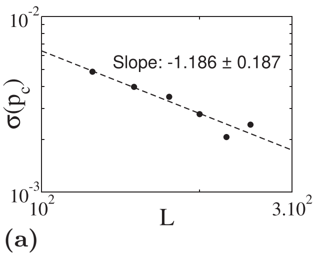

FIG. 3. Fluctuations of (a) the critical threshold $\sigma(p_c)$ and (b) the mass of the giant component at criticality $\sigma(M_c)$. The dashed lines correspond to a power-law fit to the data, and we inform the slopes and their standard deviation. The scaling exponents obtained for mean-field networks with $\zeta = L$ are consistent with those evaluated for interdependent networks in two dimensions [6], $\nu' \cong -1$ and $d_f' \cong 1.5$. Results for each value of L correspond to an average of 40 independent realizations.

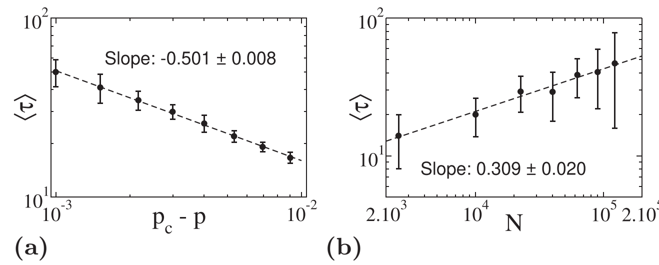

FIG. 5. Scaling behavior of the average total time of the cascade, $\langle\tau\rangle$, for $\zeta = 100$. The dashed lines correspond to a power-law fit of the curves, and we inform the slopes and their standard deviation. (a) $\langle\tau\rangle$ scales with p near criticality, $p_c - p$ $(p < p_c)$, with an exponent $\cong -0.5$ (average of ten realizations). (b) $\langle\tau\rangle$ scales with system size N, at criticality, with an exponent $\cong 1/3$ (average of 20 realizations up to $N = 90\,000$, and of 15 runs for $N = 122\,500$). These results are similar to those found for interdependent networks [21]. Error bars are included for each point of both curves.

In Fig. 4(d) we show the spatiotemporal distribution of the failures just above criticality, where failures spread at all times over the whole network. This occurs because optimal paths that disappear after some failures are likely to be replaced by paths that pass through distant nodes due to long-range connections, and then these distant nodes become overloaded.

The process for spatial networks [$\zeta = 3$, Figs. 4(e)–4(h)] is strikingly different. Since the typical length of links is short (compared to L), initial failures due to overloads may concentrate and spread radially to close neighbors [Fig. 4(h)]. Eventually, near criticality, overloads and failures create a hole of failed nodes within the functional giant component, which grows spontaneously and spreads throughout the entire system, causing its collapse. This phenomenon is known as nucleation, and it has also been observed in interdependent lattices with finite-length dependency links [20,22] and in spatial multiplex networks [10,11]. In addition, the complete disintegration of the giant component develops in a prolonged time interval with a relatively short plateau stage and a more gradual collapse [in contrast to weakly embedded systems, as seen in Fig. 4(a)].

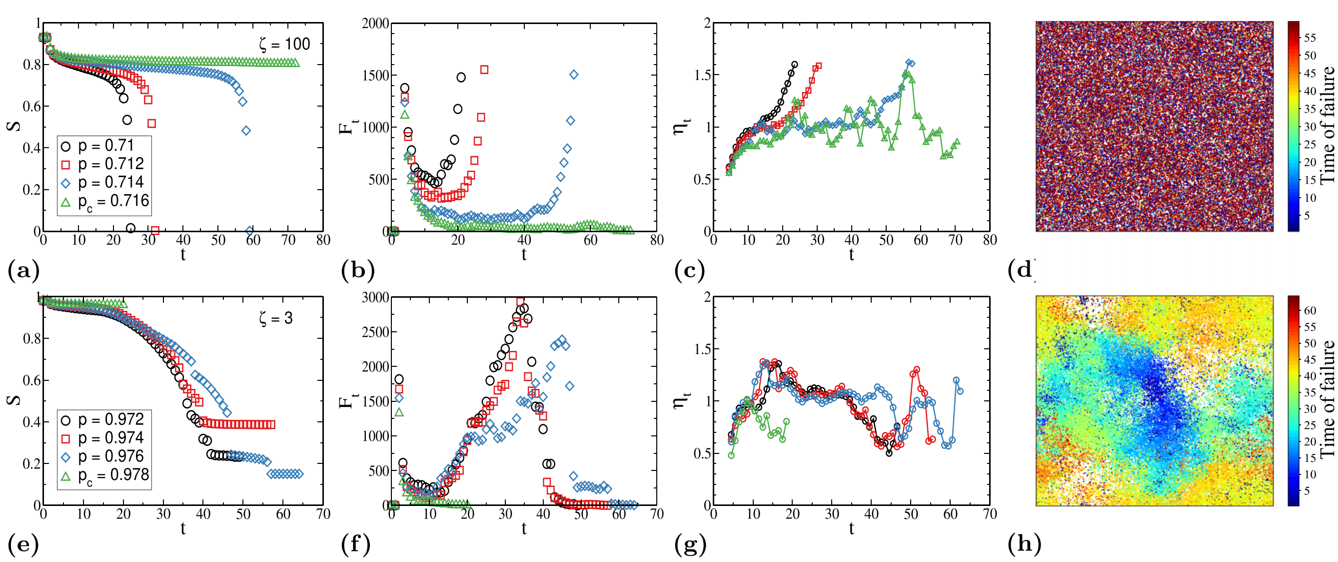

FIG. 4. Dynamic behavior of overload failures model near criticality, for networks with $\zeta = 100$ (top figures) and $\zeta = 3$ (bottom figures). From left to right: (a) and (e) Evolution of the giant component relative size S, with t the number of iterations. (b) and (f) Instant failures F_t. Note the microscopic values and the flatness increment in (b), in contrast to that in (f). (c) and (g) Moving average of the branching factor η_t. η_t stays for a long time around 1, indicating critical branching. (d) and (h) Spatiotemporal propagation of the failures. Colors represent the time of node failure close to criticality for (d) $p = 0.714$ $(\zeta = 100)$ and (h) $p = 0.976$ ($\zeta = 3$, the color map is centered to improve visualization). It is clearly seen in (h) that short-range connections in strongly embedded networks originate a spatial-radial spreading of failures, in a process known as nucleation (see also Ref. [28]). This is in contrast to (d) where damage spreads to any location due to long-range links $(\zeta = 100)$. Legends in [(a)–(c)] are the same, as well as in [(e)–(g)].

IV. DISCUSSION

In this paper we study the critical behavior and exponents that characterize the steady state and the dynamics of cascading failures due to overloads, governed by the ML model and triggered by randomly removing a fraction of links, in both strongly and weakly spatially embedded 2D networks, which have a typical link length ζ.

For the weakly or nonembedded systems we observe a usual mixed-order transition similar to that of interdependent random networks, with a critical exponent value of $\beta = 0.5$. Furthermore, fluctuations of the quantities p_c and $M(p_c)$ present exponents also in agreement with those of interdependent networks. These exponents characterize the correlation length and the fractal fluctuations of the order parameter. In contrast, strongly embedded networks do not show a curvature (singularity) in the order parameter near p_c, but rather a linear decrease, as in interdependent spatial networks, which is a characteristic of pure first-order transitions. Regarding dynamical aspects near the transition, weakly and strongly embedded systems also show a strikingly different behavior. Studying the spatiotemporal propagation of failures, we find that for large ζ the failures spread through the whole network at all times. In contrast, for small ζ, initial failures are likely to initiate in a random location and propagate to nearby sites, yielding to a nucleation spreading process that is observed as well in spatial interdependent and multiplex networks [20] (see also the recent study by Choi *et al.* [29]).

Our results regarding the temporal evolution of CFO and those corresponding to critical exponents at the steady state, for both small and large ζ, show a remarkable similarity to those of pure percolation (in the absence of overloads) of interdependent networks, for short- and long-range dependencies, respectively. Therefore, we suggest that the overload mechanism of failure propagation plays a similar role to that of dependencies in networks, and that both systems may belong to the same universality class.

We recognize that our study is limited to the model of cascading failures proposed by Motter and Lai, in which the shortest or optimal paths play a crucial role in determining the loads of the nodes and thus intervene in the dynamics of the process. Further research exploring diverse failure propagation models would help to extend the scope of the results found in this paper. For instance, one interesting model to analyze would be the direct current approximation of power grids, where nodes (generators, loads, or transmission nodes) satisfy the Kirchhoff equation, which will be our aim in a future study.

In addition, the time-costly numerical simulations that allowed us to carry out this study also limit us in the amount of results that we can produce in a certain amount of work time. In this way our results, which are only performed for spatial dimension $d = 2$, represent an indication that overloads in networks and interdependent networks belong to the same universality class. However, it would be interesting to explore higher dimensions and analyze the dependence of the critical exponents on d, and test whether higher dimensions show also a similar behavior to that of Ref. [6] [Figs. 2(a) and 3(a)] for the fluctuations of the critical quantities p_c and M_c. Related to this, and considering our limitation to study systems up to a certain size, an analysis on how critical exponents approach the values obtained in this work as the size of the system increases would make our results more robust.

Cascading failures due to overloads can dramatically alter the functioning of relevant infrastructures (e.g., electrical power grids, and communication and transportation networks). Thus, researchers from various disciplines are interested in developing a well-founded framework for understanding how such catastrophic processes behave and what are their microscopic origin. We find that when long-range links appear the cascade of failures occurs throughout the system, while for short-range links (with respect to the system size) a nucleated damage occurs and propagates radially throughout the system. Our study could therefore be useful for devising and building more resilient infrastructures, in order to avoid or mitigate such catastrophic breakdowns.

ACKNOWLEDGMENTS

I.A.P., C.E.L. and L.A.B. wish to thank to UNMdP (EXA 1056/22), FONCyT (PICT 1422/2019) and CONICET, Argentina, for financial support. S.H. wishes to thank the Israel Science Foundation (Grant No. 189/19), the Binational Israel-China Science Foundation (Grant No. 3132/19), the NSF-BSF (Grant No. 2019740), the EU H2020 project RISE (Project No. 821115), the EU H2020 DIT4TRAM, and the EU H2020 project OMINO (Grant No. 101086321) for financial support. This research was supported by a grant from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel (Grant No. 2020255).

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