

## Comment on “Inverse Square Lévy Walks are not Optimal Search Strategies for $d \geq 2$ ”

It is widely accepted that “inverse square Lévy walks are optimal search strategies because they maximize the encounter rate with sparse, randomly distributed, replenishable targets” [1] when the search restarts in the vicinity of the previously visited target, which becomes revisitable again with high probability, i.e., nondestructive foraging [2]. Three objecting claims are raised in Ref. [1] for  $d \geq 2$ : (i) the capture rate  $\eta$  has linear dependence on the target density  $\rho$  for all values of the Lévy index  $\alpha$ ; (ii) “the gain  $\eta_{\max}/\eta$  achieved by varying  $\alpha$  is bounded even in the limit  $\rho \rightarrow 0$ ” so that “tuning  $\alpha$  can only yield a marginal gain”; (iii) depending on the values of the radius of detection  $a$ , the restarting distance  $l_c$ , and the scale parameter  $s$ , the optimum is realized for a range of  $\alpha$ .

We agree with claim (i), but as we will see, it is not relevant in  $d \geq 2$  to whether or not inverse square Lévy walk searches are optimal for nondestructive foraging. Claim (iii) is also correct, however, this claim was made already in Refs. [2–5]. In particular, Ref. [2] showed that  $\alpha = 1$  is optimal only in the limit  $l_c \rightarrow a$ , which is the main condition of nondestructive foraging, with the quantity  $l_c$  in Ref. [1] being none other than the distance  $r_o$  in Ref. [2]. Otherwise for large  $l_c$  the optimal strategy in the limit  $\rho \rightarrow 0$  is to go along straight lines, i.e.,  $\alpha \rightarrow 0$ . Moreover, it has been known since 2003 that a range of  $\alpha$  can be optimal (see Fig. 1 of Ref. [3], Figs. 2–3 of Ref. [4], and Figs. 1 and S1 of Ref. [6], none of which are cited in Ref. [1]). Crucially, claims (i) and (iii) do not *per se* contradict the main finding of Ref. [2] that  $\alpha = 1$  is optimal under the specific conditions of nondestructive foraging (or of destructive foraging in patchy landscapes) [2–10].

To test claim (ii), we have numerically simulated the identical model proposed in Ref. [1] (see Fig. 1). The scaling for  $\eta$  with  $\rho$  proposed in Ref. [2] and proved in Ref. [8] for  $d = 1$  does not hold in  $d = 2$ , in agreement with Ref. [1]. However, we find, for small enough  $\delta = l_c/a - 1$ , that  $\eta$  develops a maximum at  $\alpha = 1$  with an arbitrarily large gain relative to the ballistic ( $\alpha \rightarrow 0$ ) and Brownian ( $\alpha = 2$ ) limits, contradicting claim (ii) about “marginal gain” in Ref. [1].

The main problem with Ref. [1] is that Eq. (3) fails in the limit  $l_c \rightarrow a$  of nondestructive foraging. Equation (3) yields a gain  $K_d \sim 1/[A(a^\beta - Bl_c^\beta)]$  in Eq. (5), with  $\beta = -1$  for  $\alpha < 1$  and  $\beta = \alpha - 2$  for  $\alpha > 1$ . This gain, which agrees with claim (ii), is wrong in the limit  $l_c \rightarrow a$ .

Finally, we present a heuristic argument for the correct scaling of  $K_d$  for  $d = 2$  when  $l_c \rightarrow a$ . Note that  $l_c$  is the distance at which the target stops hiding. The limit  $\delta \rightarrow 0$  has biological relevance in this “hide-and-seek” model [10]. Let  $\sigma = s/a$  and  $\eta_0(\alpha, \delta, \rho, \sigma) = \eta/(\rho a)$ . When  $\delta \rightarrow 0$ , the (radial) motion of the forager near the border of the detection circle is essentially one dimensional, hence

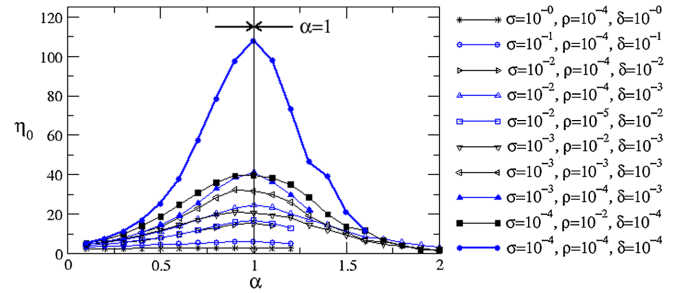


FIG. 1.  $\eta_0 = \eta/(\rho a)$  vs  $\alpha$  for  $N = 10^6$  Poisson distributed targets on a square of size  $\sqrt{N/\rho}$  with periodic boundary conditions averaged over  $10^5$  targets found.

the rigorous theory of the Riesz operator [8] on the interval of length  $L$  with absorbing ends becomes applicable. For  $\sigma > \delta$ , the efficiency increases when  $\sigma$  decreases because there are fewer large jumps leading away from the previous target that make reencountering it difficult. When  $\sigma \approx \delta$ , the efficiency reaches its maximum. In the limit  $\sigma \approx \delta \rightarrow 0$ , we expect the same scaling behavior as in  $d = 1$ :  $\eta_0 \sim \delta^{-\alpha/2}$  for  $\alpha < 1$  and  $\eta_0 \sim \delta^{-1+\alpha/2}$  for  $\alpha > 1$ . Hence,  $\eta_0$  has an arbitrarily strong maximum at  $\alpha = 1$  when  $\sigma \approx \delta \rightarrow 0$  in agreement with Fig. 1 and in disagreement with the title and claim (ii) of Ref. [1], restoring thus the original result for nondestructive foraging in Ref. [2] of the optimality of inverse square Lévy flights.

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