Dynamics of Critical Cascades in Interdependent Networks

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The failure of interdependent networks, as well as similar avalanche phenomena, is driven by cascading failures. At the critical point, the cascade begins as a critical branching process, where each failing node (element) triggers, on average, the failure of one other node. As nodes continue to fail, the network becomes increasingly fragile, and the branching factor grows. If the failure process does not reach extinction during its critical phase, the network undergoes an abrupt collapse. Here, we implement the analogy between this dynamic and birth-death processes to derive new analytical results and significantly optimize numerical calculations. Using this approach, we analyze three key aspects of the dynamics: the probability of collapse, the duration of avalanches, and the length of the cascading plateau phase preceding a collapse. This analysis quantifies how the system size and the intensity of the initial triggering event influence these characteristics.

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Network science has experienced remarkable development in recent decades, finding extensive applications across various fields, including the study of neural, social, metabolic, and communication networks [1,2]. Within this domain, the topic of interdependent networks has gained significant attention in recent years [3]. This concept has also found a wide range of applications, from critical infrastructures and cybersecurity to the dynamics of ecological communities [4–6]. Recent work has effectively demonstrated the key features of interdependent network collapse using a clear and physically straightforward experimental model, which involves thermally connected layers of amorphous superconductors [7].

Interdependent networks are characterized by two types of links: *connectivity* links within each network and *dependency* links between nodes of different networks. Each link type serves a distinct role. Connectivity links are understood as in standard percolation problems, so nodes that disconnect from the giant connected component (GCC) of their network are considered nonfunctional. Dependency links, on the other hand, propagate damage between the networks: if a node in one network fails, its dependent node in the other network will fail as well, even if it is still connected to the GCC of its own network [see Fig. 1(a)]. When nodes are removed from one network, the interplay between these two types of links can trigger a cascading failure. Therefore, while a single network experiences a continuous second-order transition in which the GCC collapses when $1 - p_c$ of the nodes are removed, interdependent networks undergo an abrupt, first order transition, at $\tilde{p} > p_c$ [Fig. 1(b)] [3].

These cascading failures are a general characteristic of avalanche phenomena [8], which have also been extensively studied in recent decades in various contexts such as magnetic materials, plastic distortions, crack propagation, earthquakes, neural networks, and so on [9–13]. In all of these systems, a failure or change in the state of a single component, such as a spin, imposes "pressure" on adjacent components, and a critical point in parameter space separates localized avalanches from widespread collapses. Such cascades are mathematically analogous to stochastic birth-death processes, where individuals can produce a random number of offspring [11,14–17].

In this Letter, we aim to examine the analogy between demographic processes and the dynamics of cascading failures. In particular, we will focus on the critical point, where the corresponding dynamics begins as a neutral birth-death process [18–21] but, over time, diverges from neutrality. We will implement this analogy to derive novel qualitative and quantitative insights.

Specifically, our study addresses three crucial questions. First, how does the likelihood of a collapse change in relation to the scale of the initial attack? Second, what is the impact of this initial attack size on the "plateau time"—the period of slow dynamics preceding the collapse [Fig. 1(c)]? Third, how does the initial attack size impact the mean "avalanche time," the duration until an avalanche ends, whether or not it causes collapse?

From nodes to generations—As depicted in Fig. 1(a), damage cascades reciprocally between dependent networks, with each failed node having a chance $P_{1\to m}$ of triggering the failure of m nodes within the same network in

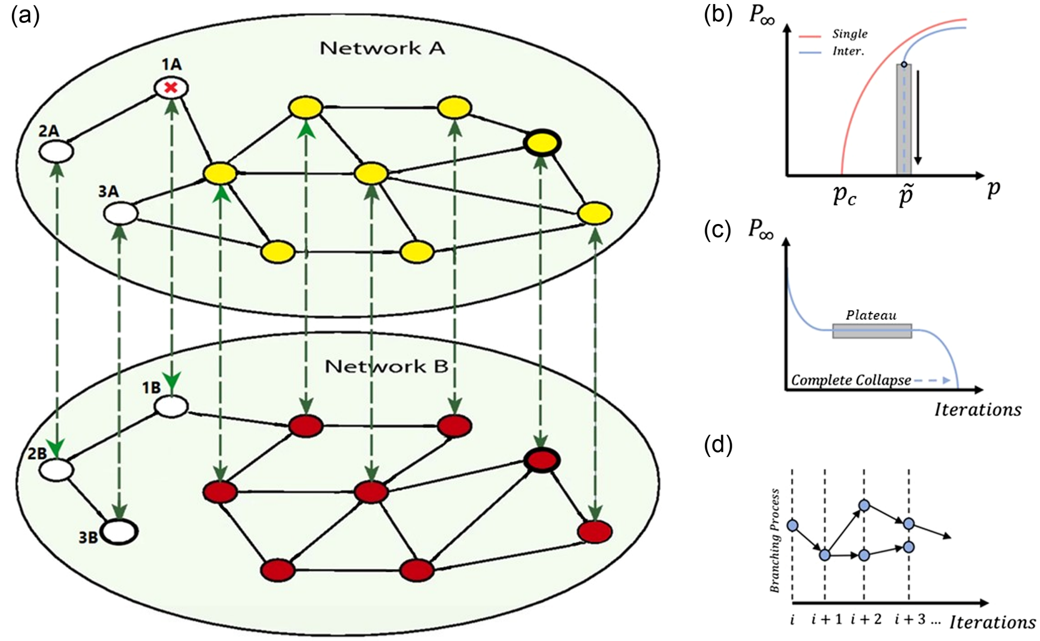

FIG. 1. Cascading failures in interdependent networks conceptualized as a birth-death process. Panel (a): two networks, A and B, interconnected by dependency links (green) alongside internal connectivity links (black). When a node fails, it topples down (in the same network) all the nodes it disconnects from GCC. Additionally, each failed node also causes the dependent node in the parallel network to fail, leading to a cascading effect. For instance, the failure of node 1A (marked with a red X) results in the detachment of 2A from the GCC, causing the failure of nodes 1B and 2B due to dependency. Consequently, 3B becomes detached from the GCC of network B, which cascades back to topple 3A. Thus the failure of 1A leads to the failure of 3A in the subsequent iteration. Panel (b): while single networks exhibit a continuous phase transition (red) in which the likelihood of a node being part of the GCC (P_∞) continuously diminishes to zero, interdependent networks undergo an abrupt transition (blue) due to cascading failures. At $p = \tilde{p}$ [panel (c)], the cascade of failures involves a long-lasting plateau, where damage stays relatively constant over several iterations. Panel (d): the cascading failure dynamics mirrors a birth-death process, wherein each failed node [like 1A in panel (a)] has a given number of offspring (like 3A) in the subsequent iteration. When $p > \tilde{p}$ the mean offspring count, $\bar{m}$, is less than one, resulting in an exponential decay of the demographic process and network survival. Conversely, for $p < \tilde{p}$, $\bar{m} > 1$, leading to exponential damage growth and a swift collapse. At $p = \tilde{p}$, $\bar{m} = 1$ during the plateau, and the dynamics is neutral.

the subsequent iteration. With a sufficiently large number of nodes, N, and a small number of failed nodes, failures are uncorrelated. Consequently, at the tth step, if the count of failed nodes is n_t, then n_{t+1}—the tally of failed nodes in the subsequent step in the same network—is the sum of n_t independent variables, following a probability distribution function of $P_{1\to m}$ (see Supplemental Material I [22]). Hence, this branching process mirrors a demographic dynamics, where nodes represent individuals and iterations are equivalent to generations [Fig. 1(d)]. At $p = \tilde{p}$, the process is *neutral*, meaning that, during the plateau [Fig. 1(c)], the mean number of offspring per node is one.

The "population dynamics" of n_t is influenced by demographic stochasticity, namely the binomial noise associated with the randomness of the birth-death process, and deterministic forces that yield exponential growth (if $\bar{m} > 1$) or exponential decay (if $\bar{m} < 1$). At $\tilde{p}$ the growth rate vanishes, and the dynamics is neutral: the number of failures in each generation, n, is (on average) time independent, and its time evolution is governed solely by demographic noise. Because $n \ll N$, the GCC undergoes a prolonged plateau, in which P_∞ is almost fixed [Fig. 1(c)]. This plateau phase may end in extinction, in which case $n_t = 0$ and the network survives. However, if the process persists long enough, the GCC becomes thinner and more fragile, $\bar{m}$ increases above one, the population of failed nodes starts growing faster, and the network collapses.

Let us provide an approximate expression for the deviation from neutrality as the process goes on. t is defined as the number of iterations since the beginning of the plateau, so at $t = 0$ the chance to be on the infinite cluster is $P_\infty(t = 0)$, the number of failed nodes is n_0, and the mean number of offspring $\bar{m}(t = 0) = 1$. The number of nodes that fail in the tth iteration is n_t, and $M_t = \sum_{k=0}^{t} n_t$ is the total number of nodes that fail until t. Therefore, $P_\infty(t) = P_\infty(0) - M_t/N$. As long as $M_t \ll N$ we can implement the linear approximation and assume $\bar{m}(t) = 1 + CM_t/N$, where $C \equiv d\bar{m}/dP_\infty$ evaluated at $P_\infty(t = 0)$.

To simulate such a demographic process, we start with initial values $n_0 = M_0$. For each individual in n_0, we randomly draw a number m_i of descendants for the

ith individual from a truncated power-law distribution with mean $1 + CM/N$ (see Supplemental Material III for full details). In the next iteration, the population is $n_1 \equiv \sum_{i=1}^{n_0} m_i$. We then update M_0 to $M_1 = M_0 + n_1$ and iterate the process until it either terminates (i.e., when $n_t = 0$) or the network collapses, which can be approximated by the condition $M_t = (\tilde{p} - p_c)N$. Such simulations are significantly simpler and faster than emulating dependent networks and their dynamics. Monitoring the giant component of a random network is computationally costly, requiring the tracking of all node and edge connections at each step. By contrast, simulating the dynamics of a neutral population only involves drawing n random deviates at each step.

In order to demonstrate the effectiveness of our approach, we will use it to understand how the collapse dynamics depend on the strength of the initial attack, that is, on the number of nodes that failed in the first step, n_0. Any such attack can end in one of two ways: either the GCC in both networks still exists, in which case the attack is considered a failure, or the GCC collapses, in which case the attack is considered successful.

In general, the analysis of such processes involves three fundamental questions. First, what is the probability of success? Second, what is the average number of iterations until success, assuming that the attack ended successfully? Third, what is the average number of iterations until the attack is over, where in this third case we do not distinguish between successful and failed attacks? At the critical point $p = \tilde{p}$, all three of these quantities depend on the size of the network N and on the strength of the initial attack n_0. As will be shown, the dependence on n_0 enters through the variable n_0^3/N.

Let us start by comparing two types of numerical results: those obtained from a direct simulation of interdependent networks, a complex simulation that requires building a suitable network and tracking the size of the mutual GCC at each step, versus the simple and fast simulation of the corresponding demographic process.

For direct simulations, we prepared two interdependent networks at their critical state. At the critical point, the dynamics within each network is such that, on average, each failing node triggers the detachment of *one* other node from the GCC. Accordingly, we identified the corresponding value of $\tilde{p}$ by analyzing a single network. Then we duplicate it and connect the nodes of the two resulting networks via randomly chosen dependency links. See Supplemental Material IV.

Once we have a system at criticality, we can examine the characteristics of the collapse process and compare it with the results obtained for the demographic process as described above.

Figure 2 demonstrates the power of our technique. For the same n_0, we examined the mean time until collapse T_F and the mean time to absorption T_A and compare the demographic process (where time is measured in generations) with the collapse of the interdependent networks (where time is measured in iterations). The qualitative and quantitative match is very good, which demonstrates that the demographic model indeed reflects the dynamics of the networks.

After establishing a quantitative correspondence between the two processes, we can apply well-known

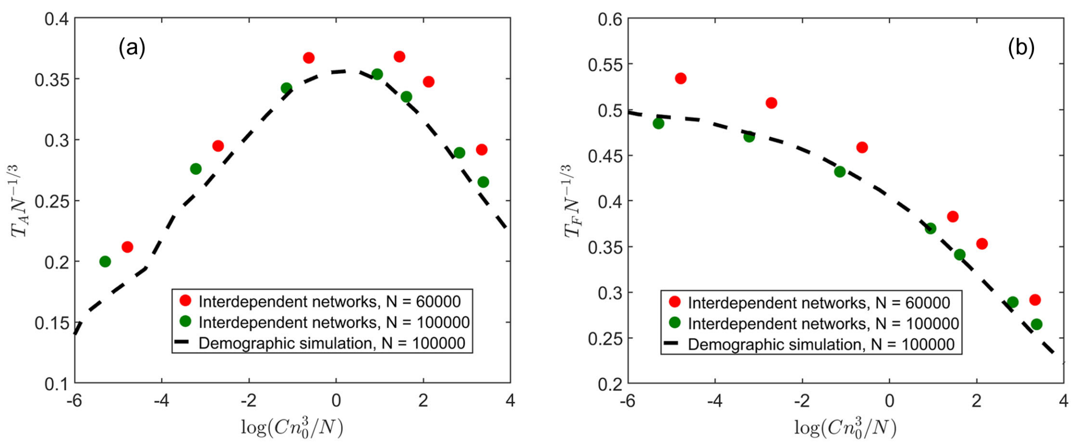

FIG. 2. Comparison between the dynamics of interdependent networks and the correspondent demographic process. Panel (a): the mean time to absorption T_A (time until the cascading failure process ends, either in a collapse of the network or in its survival). Panel (b): the mean time to failure T_F (time until the cascading process ends in a failure, conditioned on failure). In both panels, markers present the outcome of an explicit interdependent networks dynamics, whereas the dashed lines show the results of the corresponding demographic process. We used Erdös–Rényi networks with average degree $k = 5$, and the results present an average over 10^4 different network realizations for each N. Note that the larger the network is, the better the fit to the simulations of demographic theory is. For additional details and metrics regarding the distribution of outcomes, see Supplemental Material II. The basic scaling with $N^{1/3}$, implied in these graphs, is derived following Eq. (5).

techniques from the neutral theory [18,23] to derive analytical solutions for interdependent networks problems. To begin, let us consider $\Pi(n_0, N)$, the probability of collapse given n_0 and N. The analogous problem in neutral dynamics is the probability of ultimate fixation.

Switching to continuous time, the Langevin equation for a population under pure demographic stochasticity is $dn/dt = \eta(t)\sqrt{n}$, where $\eta(t)$ is a white noise process [24]. As explained, at criticality the cascading failure process corresponds to the dynamics of a population for which the growth rate is $CM(t)/N$, where $M(t)$ is the accumulated number of birth (failed nodes) until t and C is a constant. Accordingly, the coupled Langevin equations characterizing the interdependent network process are delineated as follows:

$$\frac{dn(t)}{dt} = \frac{CM(t)}{N}n + \eta(t)\sqrt{n}, \qquad \frac{dM(t)}{dt} = n. \quad (1)$$

As long as the process is neutral (i.e., when M/N is small), a process that survives until t admits $n(t) \sim t$ (see Supplemental Material V for further details) and $M(t) \sim t^2$. Therefore, $M \sim n^2$, and hence Eq. (1) may be approximated as

$$\frac{dn(t)}{dt} = Cn^3/N + \eta(t)\sqrt{n}. \quad (2)$$

The corresponding backward Kolmogorov equation for the chance of network collapse [23], Π, as a function of the initial damage n_0, can be written for the process described in Eq. (2) as

$$n\Pi''(n) + C\frac{n^3}{N}\Pi'(n) = 0, \quad \Pi(0) = 0, \quad \Pi(N^*) = 1, \quad (3)$$

where $N^* \equiv N(\tilde{p} - p_c)$. The solution is

$$\Pi(n_0) = 1 - \Gamma(1/3, z/3)/\Gamma(1/3), \quad (4)$$

where $z = Cn_0^3/N^*$ and $\Gamma(a,x) = \int_x^\infty t^{a-1}\exp(-t)dt$.

In Fig. 3 we see a comparison between this solution and the chance that two interdependent networks will collapse, as obtained from direct simulations.

Another immediate result of Eq. (2) is the general scaling behavior at criticality. The Langevin equation for a stochastic population and growth rate s is given by [23]

$$\frac{dn(t)}{dt} = sn + \eta(t)\sqrt{n}. \quad (5)$$

Comparing Eq. (5) with Eq. (2) one can see that s is analogous to Cn^2/N. In processes described by Eq. (5), the stochastic component becomes negligible (i.e., the process becomes, effectively, deterministic exponential growth) at $n \approx 1/s$ [25]. Equivalently, the transition for interdependent networks takes place at $n \approx N/Cn^2$. Therefore, the basic scaling of the cascading collapse process is $n^3 \sim T$; since under neutral dynamics $n \sim T$, we obtain $T \sim N^{1/3}$, in agreement with the simulation results of [17] and the scaling argument in [26].

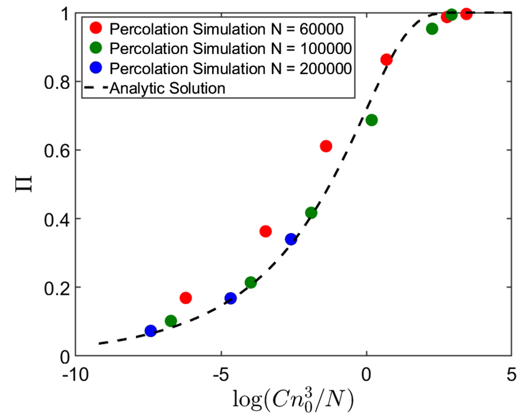

FIG. 3. Chance of collapse, Π, as a function of the initial damage. The dashed line shows $\Pi(z)$ as a function of $z = Cn_0^3/N$, as predicted in Eq. (4), for the chance of a demographic process that satisfies Eq. (2) to reach fixation at $n = N$. Each circle corresponds to the chance of collapse of interdependent networks for a given N (as indicated in the legend) and n_0, as obtained from an average over 10^3 random initial conditions.

Discussion—In large systems, cascading failure processes can be described as branching processes [14], where each failed element has a certain probability of causing additional elements to fail. This situation is analogous to a demographic process in which each individual has a probability of producing a certain number of offspring. The process becomes critical, or “neutral,” when the average number of offspring per element equals one.

In a neutral process, the probability of an individual lineage surviving indefinitely approaches zero over infinite time. Since individuals are uncorrelated, for any finite number of initially failed elements, the cascading failure must eventually stop if the system is large enough. However, in the case of interdependent networks, the system becomes increasingly fragile as the number of failed components grows. Therefore, if the process persists long enough, it will become supercritical, and the network will collapse.

The key question is how fragility increases relative to the extent of the damage. In our case, the deviation from criticality is determined by the ratio between the number of failed components and the system size. Consequently, the time during which the process remains approximately neutral depends on N and, in our case, scales as $N^{1/3}$. If the damage required for a substantial deviation from criticality does not depend on N, one observes a standard nucleation process, where the amount of damage required to ensure transition to the propagative phase does not scale with the system size.

It is noteworthy that, overall, our avalanche process resembles the spread of epidemics at their critical point [27–29]. In both cases, as long as the total number of infected individuals remains much smaller than the population size N, the process is entirely neutral. Furthermore, as in our case, when the demographic process persists long enough and the number of infected individuals scales with N to some power, the dynamics become non-neutral. The key difference between avalanches and epidemics lies in the *sign* of the effect. In ecological demographic processes, the number of offspring per individual typically *decreases* as population size increases, leading to a continuous, second-order transition. For example, in infection processes, as more individuals become infected, fewer remain susceptible. By contrast, in interdependent networks, the opposite effect occurs: as more nodes fail, the number of offspring (failures) increases, resulting in a discontinuous, first-order collapse. The relevant equation [Eq. (2)] describes a process that reaches an infinite population in finite time, so the central question is whether the neutral process can survive long enough for $M(t)$ to reach a level where significant deterministic growth occurs.

This type of dynamics, where the failure of one component triggers a cascading failure, is not unique to interdependent networks. It appears in many avalanchelike processes, such as fracture dynamics, magnetic systems, and socioecological transitions [9,10,16,30]. In all of these systems, criticality is defined by each failing element, on average, triggering the failure of one additional element. Moreover, while our analysis focuses on Erdős–Rënyi networks with low clustering, this defining criterion suggests that the same critical behavior should emerge in other network types as well. In other network architectures (e.g., scale-free networks) the variance in the number of triggered failures may be higher, potentially making them more robust. Testing this hypothesis, however, requires further study.

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