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External field and critical exponents in controlling dynamics on complex networks

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E-mail: hillel.sanhedrai@gmail.com**Keywords:** network dynamics, complex networks, phase transitions, critical exponentsSupplementary material for this article is available [online](#)

Abstract

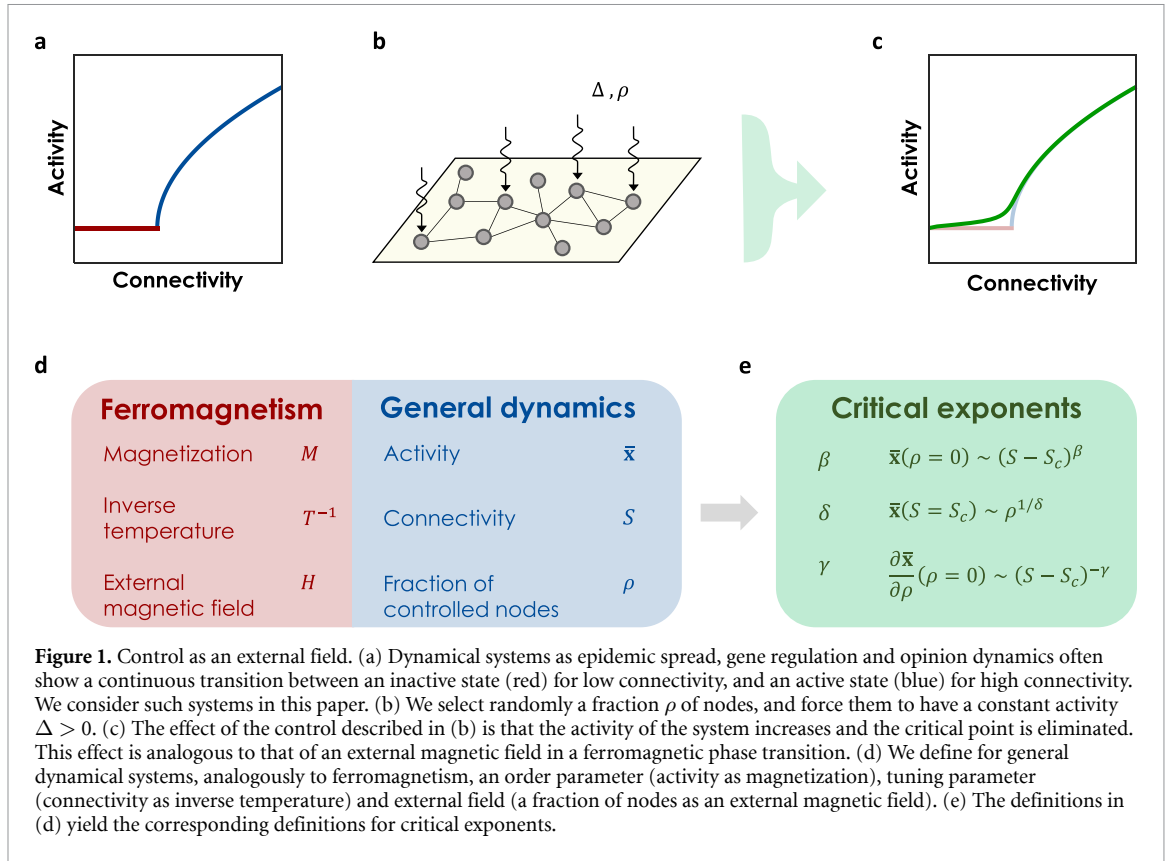
Dynamical processes on complex networks, ranging from biological, technological and social systems, show phase transitions (PTs) between distinct global states of the system. Often, such transitions rely upon the interplay between the structure and dynamics that takes place on it, such that weak connectivity, either sparse network or frail interactions, might lead to global activity collapse, while strong connectivity leads to high activity. Here, we show that controlling dynamics of a fraction of the nodes in such systems acts as an external field in a continuous PT. As such, it defines corresponding critical exponents, both at equilibrium and of the transient time. We find the critical exponents for a general class of dynamics using the leading orders of the dynamic functions. By applying this framework to three examples, we reveal distinct universality classes.

1. Introduction

Phase transitions (PT) have attracted tremendously broad research ranging from states of matter through superconductivity to ferromagnetism and many other systems [1–3]. Particularly, a special focus has been given to the behavior near the transition, revealing various critical phenomena including critical exponents and universality classes [1–3]. In the field of complex networks, a percolation PT has been widely explored where the order parameter is the relative size of the giant component while the tuning parameter is the occupation probability [4–8]. In this PT, the network is regarded as failed only when it completely *loses* its connectivity.

Here, analogous to percolation, we consider the PT of dynamical complex systems caused by structural variation of the network [9–13], where a too *weakly connected* network, even if still connected, does not function normally as if it is strongly connected. Examples for such systems are gene regulation [14], ecological networks [9], epidemics [15–17], opinion dynamics [18], and more. Therefore, in this context, the activity state of the system is regarded as the order parameter, and the tuning parameter is the connectivity as we define below. A relevant element of such systems is the impact of controlling nodes' activity on network dynamics. This matter has been investigated from several aspects, both empirical as manipulations of gene expression [19] and theoretical, ranging from controllability theory [20–22], through propagation patterns of small perturbations across the network [23–25], to global effects on the system state [26–29].

In this paper, we focus on the critical behavior of the above dynamical systems, and we show that the external intervention in dynamics of a fraction of nodes can be regarded as an external field in continuous PT, analogously to an external magnetic field in a ferromagnetic PT, which has been explored extensively [30]. We define, correspondingly, new critical exponents besides those which are already known [31, 32], and we find the values of the exponents for a *general form of dynamics*. By applying our general framework to several dynamical models we reveal distinct universality classes which indicate essential distinct responses to control around criticality. In addition to the critical exponents of the equilibrium, we define and analyze also critical exponents related to the transient time towards equilibrium [33, 34]. Different from equilibrium



critical exponents which can be formed by only two independent exponents, the three transient exponents are independent. Nevertheless, all exponents can be derived from three independent exponents.

2. Controlling system dynamics

To analyze the impact of an intervention in network dynamics we rely upon a general framework [23–25] to model nonlinear dynamics on networks. Consider a system consisting of N components (nodes) whose activities x_i ($i = 1, 2, \dots, N$) follow the Barzel–Barabási equation [23]:

$$\frac{dx_i}{dt} = M_0(x_i) + \omega \sum_{j=1}^N A_{ij} M_1(x_i) M_2(x_j). \quad (1)$$

The first function, $M_0(x_i)$ captures node i 's self-dynamics, describing mechanisms such as protein degradation [35] (cellular), individual recovery [17, 36] (epidemic) or birth/death processes [37] (population dynamics). The product $M_1(x_i) M_2(x_j)$ describes the i, j interaction mechanism, e.g. genetic activation [14, 38, 39], infection [17, 36] or symbiosis [40]. The connectivity matrix A captures the interactions (links) between the nodes, i.e. the network. An element A_{ij} equals 1 if there is a link (interaction) between nodes i and j and 0 otherwise. We consider here a matrix A which is symmetric and obeys the configuration model framework, that is a random network with a given degree distribution p_k . The strength of the interactions is governed by the *positive* uniform parameter ω , representing constructive or attractive interactions, excluding competitive interaction or oscillatory coupling functions, that are not discussed in this paper.

As an external intervention in network dynamics we consider the following simple control. We *force* a set of nodes, \mathcal{F} (a fraction ρ of the system), to have a constant activity value Δ (figure 1(b)), while all the rest in the complementary set, \mathcal{D} , are governed by the original dynamics. Thus, such a forced system obeys the set of equations,

$$\begin{cases} x_i = \Delta & i \in \mathcal{F}, \\ \frac{dx_i}{dt} = M_0(x_i) + \omega \sum_{j=1}^N A_{ij} M_1(x_i) M_2(x_j) & i \in \mathcal{D}. \end{cases} \quad (2)$$

In this study, we assume that the set of controlled nodes, \mathcal{F} , is selected randomly. Next, we aim to track the states of the unforced nodes, i.e. the set \mathcal{D} .

Using a mean field approximation [9, 28] (see SI section 2), and considering a *random* selection of controlled nodes, we obtain for the steady states [28],

$$\mathcal{S} = \frac{-M_0(\bar{\mathbf{x}})}{(1 - \rho)M_1(\bar{\mathbf{x}})M_2(\bar{\mathbf{x}}) + \rho M_1(\bar{\mathbf{x}})M_2(\Delta)}, \quad (3)$$

where the order parameter, $\bar{\mathbf{x}}$, indicating the system state, is the average activity over all the neighbors within the dynamic set, \mathcal{D} , defined by:

$$\bar{\mathbf{x}} = \frac{1}{|\mathcal{D}|\langle k_{\mathcal{D} \rightarrow \mathcal{D}} \rangle} \sum_{j \in \mathcal{D}} k_j^{\mathcal{D} \rightarrow \mathcal{D}} x_j. \quad (4)$$

The quantity $k_j^{\mathcal{D} \rightarrow \mathcal{D}}$ denotes the number of free neighbors, i.e. within \mathcal{D} , of a free node $j \in \mathcal{D}$. The connectivity, \mathcal{S} , in equation (3), is defined as:

$$\mathcal{S} = \omega \kappa, \quad (5)$$

where ω is the interaction strength from equation (1), and κ is the average neighbor degree over the whole network, $\kappa = \langle k^2 \rangle / \langle k \rangle$. Finally, $\rho = |\mathcal{F}|/N$ is the fraction of controlled nodes, and Δ is the value of forcing, equation (2).

Equation (3) provides a relation between the system state, $\bar{\mathbf{x}}$, and the connectivity \mathcal{S} . By substituting $\rho = 0$ we get the phase diagram of the free system. In figure 1(a) we show a typical result of equation (3) with $\rho = 0$ which we discuss in this paper. The obtained curve exhibits a continuous PT between an inactive (red, $\bar{\mathbf{x}} = 0$) and an active (blue, $\bar{\mathbf{x}} > 0$) states for a *free* system without any external control.

3. External field

Forcing a system as described in equation (2) and illustrated in figure 1(b), with a positive value $\Delta > 0$, we obtain the typical phase diagram presented in figure 1(c) which is constructed by equation (3) with $\rho > 0$. One can see that controlling the system acts as an external field in a continuous PT [1, 8, 41]. It makes the curve smooth and eliminates the PT (green curve). Therefore, the system stable state, $\bar{\mathbf{x}}$, the connectivity, \mathcal{S} , and the fraction of controlled nodes, ρ , in PT of dynamical systems, are being an analogy of magnetization, inverse temperature and external magnetic field respectively in ferromagnetic PT, figure 1(d). Hence, we define the fraction of forced nodes, ρ , as the strength of the external field in our problem. Correspondingly, we define the following critical exponents,

$$\bar{\mathbf{x}}(\mathcal{S} \rightarrow \mathcal{S}_c^+, \rho = 0) \sim (\mathcal{S} - \mathcal{S}_c)^{\beta}, \quad (6)$$

$$\bar{\mathbf{x}}(\mathcal{S} = \mathcal{S}_c, \rho \rightarrow 0) \sim \rho^{1/\delta}, \quad (7)$$

$$\chi(\mathcal{S} \rightarrow \mathcal{S}_c^+, \rho = 0) \sim (\mathcal{S} - \mathcal{S}_c)^{-\gamma}, \quad (8)$$

where the susceptibility, χ , is defined by:

$$\chi = \left. \frac{\partial \bar{\mathbf{x}}}{\partial \rho} \right|_{\rho=0}. \quad (9)$$

In the next section, we show that these scaling relations are satisfied, and we find the critical exponents generally for given dynamics captured by the functions $M_{0,1,2}$ in equation (1).

4. General derivation of the critical exponents

We are interested in finding the critical exponents for the general dynamic functions $M_{0,1,2}$ appearing in equation (1). As aforesaid, we consider dynamics which have a stable state $\bar{\mathbf{x}} = 0$ below criticality. We also consider a continuous PT, such that the active ($\bar{\mathbf{x}} > 0$) state approaches zero when $\mathcal{S} \rightarrow \mathcal{S}_c^+$, thus we analyze

equation (3) in the limit of $\bar{x} \rightarrow 0$. To this end, we assume that the dynamical functions have Hahn expansions as power series [42]:

$$\begin{aligned}
 M_0(x) &= \sum_{n=0}^{\infty} a_n x^{\Gamma_n}, \\
 M_1(x)M_2(x) &= \sum_{n=0}^{\infty} b_n x^{\Pi_n}, \\
 M_1(x) &= \sum_{n=0}^{\infty} c_n x^{\Lambda_n},
 \end{aligned}
 \tag{10}$$

where n is an index running over all powers that appear in the expansion either the series is infinite or finite. The exponents Γ_n , Π_n and Λ_n are increasing non-negative series, and differently from the Taylor series they can be fractional.

The above functions should satisfy the characters of our problem as mentioned above, that is a continuous transition from an inactive stable state, $\bar{x} = 0$, for a weak connectivity, $\mathcal{S} < \mathcal{S}_c$, to an active stable state, $\bar{x} > 0$, for a strong connectivity, $\mathcal{S} > \mathcal{S}_c$. In addition, a controlled system should act as if under an external field. To this end (see derivation in SI section 3), the lead exponents should fulfill $\Gamma_0 = \Pi_0 > \Lambda_0$, and the lead coefficients should satisfy $a_0 < 0$, $b_0 > 0$, and $c_0 M_2(\Delta) > 0$. For the non-leading terms see SI section 4.1.

As we expand equation (3) at the limit of $\mathcal{S} \rightarrow \mathcal{S}_c^+$, $\rho \rightarrow 0$, and thus also assume $\bar{x} \rightarrow 0$, by using the expansions in equation (10), we obtain that the scaling relations of equations (6)–(8) hold, and the critical exponents depend on the lead exponents of the dynamical functions as (see SI section 4):

$$\beta = \frac{1}{m - \Gamma_0}, \tag{11}$$

$$\delta = m - \Lambda_0, \tag{12}$$

$$\gamma = \frac{m - \Lambda_0 - 1}{m - \Gamma_0}, \tag{13}$$

where

$$m = \min\{\Gamma_1, \Pi_1\}. \tag{14}$$

If Γ_1 does not exist, then simply $m = \Pi_1$, and vice versa. One can see that these three critical exponents obey the known scaling relation, the Widom’s identity, $\beta(\delta - 1) = \gamma$, implying that there are only two independent exponents which form the third one as already well-known.

In figure 2 we show simulation results that agree with our theoretical predictions for the critical exponents for epidemic, regulatory and opinion dynamics, for Erdős–Rényi random network. In section 6 we analyze specifically each dynamics and find its critical exponents summarized in table 1. As one can see, different dynamics exhibit different critical exponents which indicate they belong to distinct universality classes. The regulatory dynamics exponents depend on a parameter a . Setting a to 1 will equalize its exponents to those of epidemic dynamics. In contrast, there is no value of a that would equalize the exponents to those of opinion dynamics.

5. Transient critical exponents

After we analyzed the system state at equilibrium, we further analyze the time it takes for the system to relax towards equilibrium. To this end, we go one step back before equation (3) (see SI section 2) to use the mean-field *dynamic* equation, which follows the evolution in time of the average state of the system, [9, 28]:

$$\frac{d\bar{x}}{dt} = M_0(\bar{x}) + \mathcal{S}M_1(\bar{x})((1 - \rho)M_2(\bar{x}) + \rho M_2(\Delta)). \tag{15}$$

Similarly to the above analysis of equation (3) for finding the critical exponents of the steady state, here we analyze equation (15) in the limit of large t , $\mathcal{S} \rightarrow \mathcal{S}_c^+$ and $\rho \rightarrow 0$, and thus also $\bar{x} \rightarrow 0$, to obtain the critical exponents of the transient towards the stable state (see SI section 5 for details). Correspondingly, we define

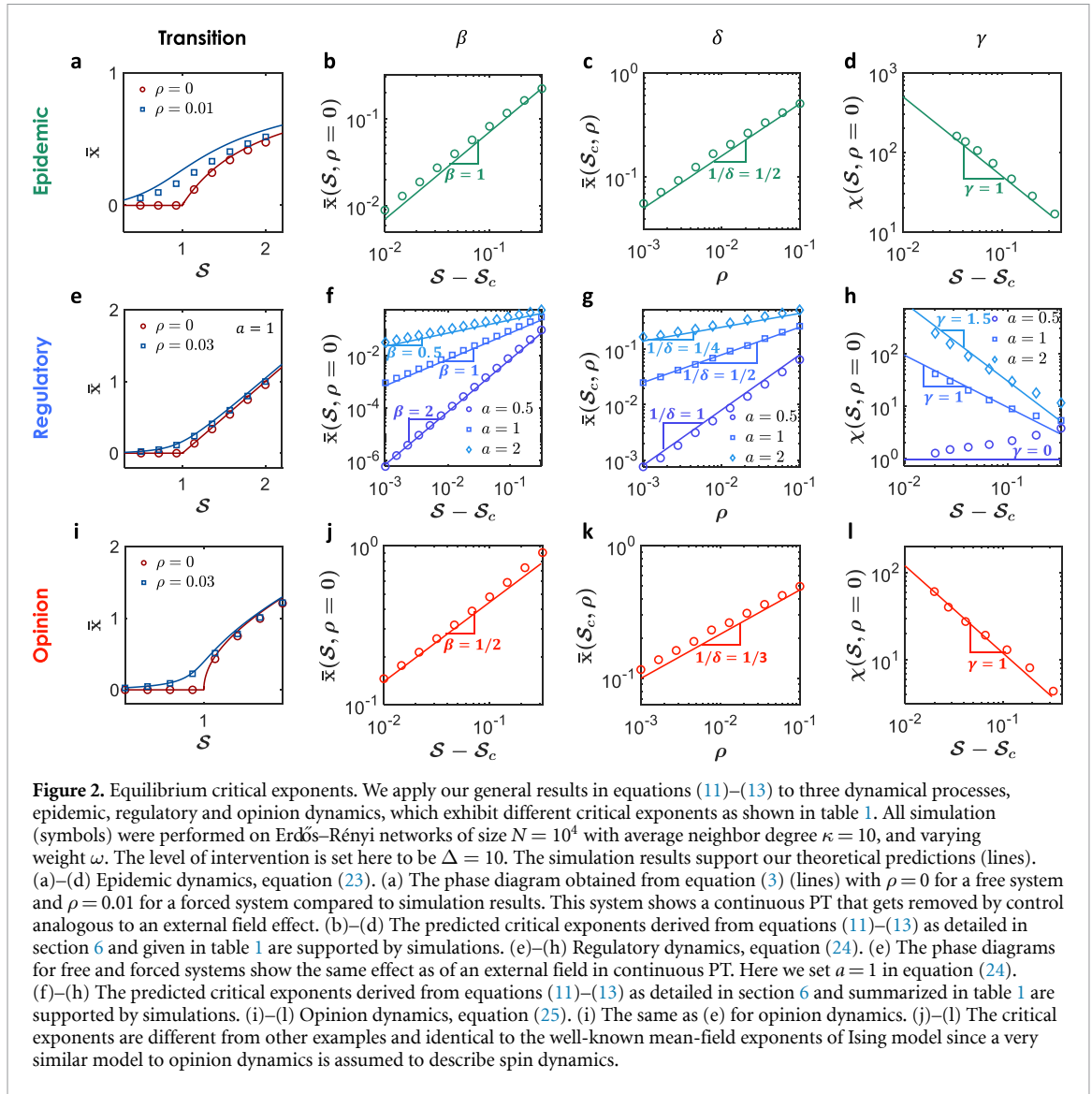


Figure 2. Equilibrium critical exponents. We apply our general results in equations (11)–(13) to three dynamical processes, epidemic, regulatory and opinion dynamics, which exhibit different critical exponents as shown in table 1. All simulation (symbols) were performed on Erdős–Rényi networks of size $N = 10^4$ with average neighbor degree $\kappa = 10$, and varying weight ω . The level of intervention is set here to be $\Delta = 10$. The simulation results support our theoretical predictions (lines). (a)–(d) Epidemic dynamics, equation (23). (a) The phase diagram obtained from equation (3) (lines) with $\rho = 0$ for a free system and $\rho = 0.01$ for a forced system compared to simulation results. This system shows a continuous PT that gets removed by control analogous to an external field effect. (b)–(d) The predicted critical exponents derived from equations (11)–(13) as detailed in section 6 and given in table 1 are supported by simulations. (e)–(h) Regulatory dynamics, equation (24). (e) The phase diagrams for free and forced systems show the same effect as of an external field in continuous PT. Here we set $a = 1$ in equation (24). (f)–(h) The predicted critical exponents derived from equations (11)–(13) as detailed in section 6 and summarized in table 1 are supported by simulations. (i)–(l) Opinion dynamics, equation (25). (i) The same as (e) for opinion dynamics. (j)–(l) The critical exponents are different from other examples and identical to the well-known mean-field exponents of Ising model since a very similar model to opinion dynamics is assumed to describe spin dynamics.

Table 1. All critical exponents found in this paper for three examples of dynamics, including the exponents related to the steady state (β , δ and γ) and also the exponents related to the transient towards relaxation (φ , ϕ and θ). The values are derived from equations (11)–(13) and (20)–(22).

Dynamics	Model	β	δ	γ	φ	ϕ	θ
Epidemic	SIS	1	2	1	1	1	$\frac{1}{2}$
Regulatory	MM	$\frac{1}{a}$	$2a$	$2 - \frac{1}{a}$	$\frac{1}{2a - 1}$	$2 - \frac{1}{a}$	$1 - \frac{1}{2a}$
Opinion	BLSS	$\frac{1}{2}$	3	1	$\frac{1}{2}$	1	$\frac{2}{3}$

additional three exponents related to relaxation time. At criticality, i.e. at $S = S_c$, $\rho = 0$, we find a power law convergence with the exponent φ defined by:

$$\bar{x}(S = S_c, \rho = 0, t \rightarrow \infty) \sim t^{-\varphi}. \tag{16}$$

This power-law convergence is valid for $m > 1$ (see SI section 5) which is the case in all our examples. When $m = 1$, the decay at criticality is exponential, and if $m < 1$, the system relaxes in a finite time.

Above criticality, that is for $S > S_c$ or $\rho > 0$, there is an exponential decay, with a decay time τ , which depends on both S and ρ :

$$\bar{x}(S, \rho, t \rightarrow \infty) \sim \exp(-t/\tau(S, \rho)). \tag{17}$$

However, when we approach the critical point, we get that the typical relaxation time τ diverges because the decay becomes as power law rather than exponential. We track two paths in (S, ρ) -space towards the critical point $(S_c, 0)$, vertical and horizontal, and define, respectively,

$$\tau(S \rightarrow S_c^+, \rho = 0) \sim (S - S_c)^{-\phi}, \quad (18)$$

$$\tau(S = S_c, \rho \rightarrow 0) \sim \rho^{-\theta}. \quad (19)$$

By expanding equation (15) close to criticality (see SI section 5), we find the transient critical exponents via the expansions of the dynamic functions, equation (10),

$$\varphi = \frac{1}{m-1}, \quad (20)$$

$$\phi = \frac{m-1}{m-\Gamma_0}, \quad (21)$$

$$\theta = \frac{m-1}{m-\Lambda_0}. \quad (22)$$

Note that these three transient exponents are independent in contrast to the above equilibrium exponents. However, all six exponents can be obtained from three independent exponents, since they all depend on only three leading terms of the dynamical functions, m , Γ_0 and Λ_0 .

In figure 3 we show the results of simulations on Erdős–Rényi network for our three dynamical examples, which exhibit good agreement with our theoretical predictions.

6. Applications

In this section, we apply our general analysis to three examples of dynamics (see also SI section 6), which fulfill our demand for a continuous PT, that behave under external control as under an external field as discussed above.

6.1. Epidemic

As our first example, we consider the susceptible-infectious-susceptible (SIS) model for epidemic spreading [15–17]. In this model, equation (1) takes the form,

$$\frac{dx_i}{dt} = -\alpha x_i + \omega \sum_{j=1}^N A_{ij}(1-x_i)x_j, \quad (23)$$

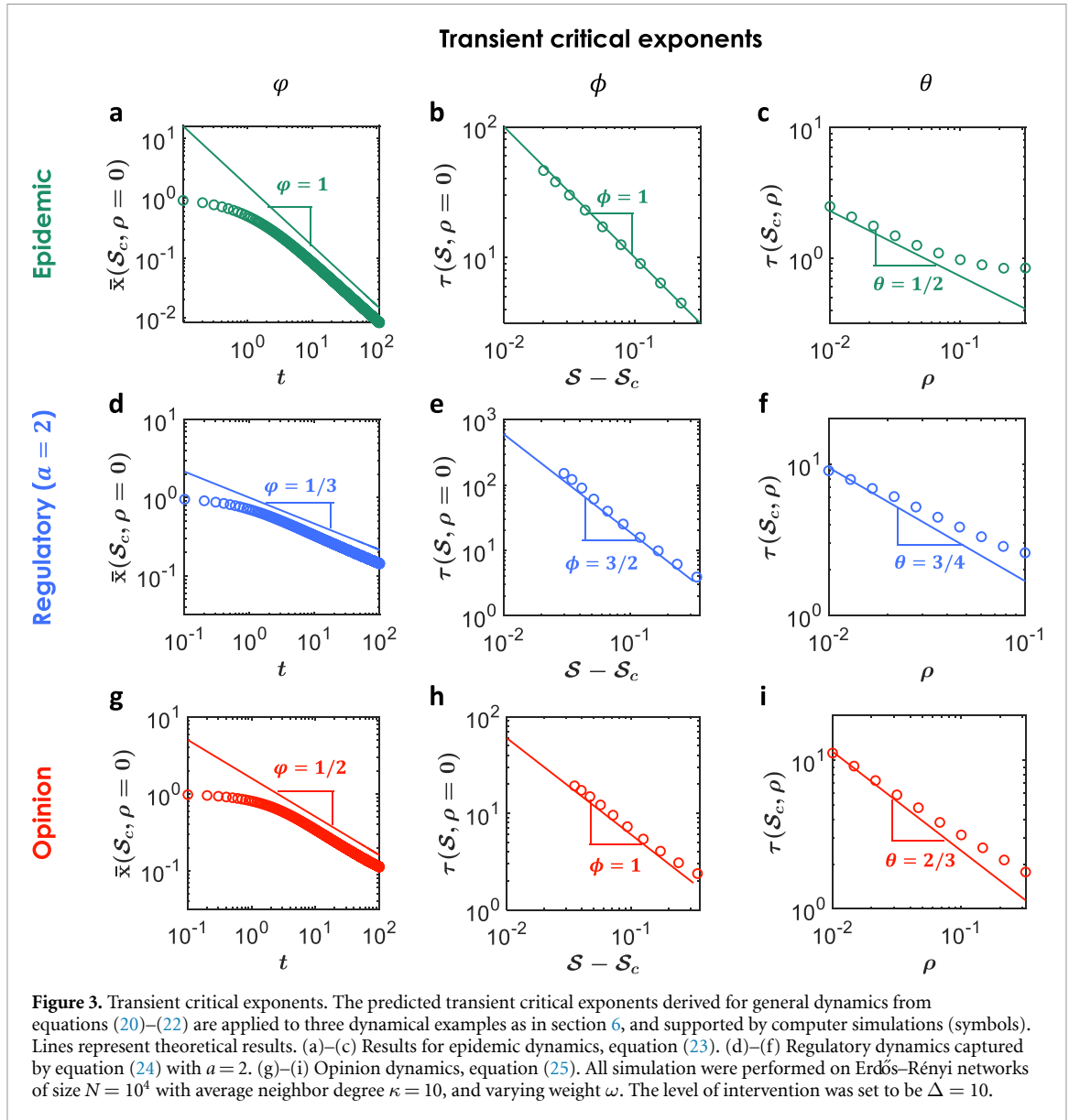
where $x_i(t)$ represents the probability of agent i to be infectious. The first term on the rhs indicates the probability of recovering with recovery rate α . We set, without loss of generality, the recovery rate to be $\alpha = 1$. The second term represents the likelihood for node i to get infected by its neighbors with rate ω . This model is mapped to our form in equation (1) by $M_0(x_i) = -x_i$, $M_1(x_i) = 1 - x_i$, and $M_2(x_j) = x_j$. Therefore, $\Gamma_0 = 1$ and Γ_1 does not exist, $\Pi_0 = 1$ and $\Pi_1 = 2$, $\Lambda_0 = 0$, and $m = 2$. Hence, the equilibrium critical exponents, using equations (11)–(13), are $\beta = 1$, $\delta = 2$, and $\gamma = 1$. The transient critical exponents, obtained by equations (20)–(22), are $\varphi = 1$, $\phi = 1$ and $\theta = 1/2$. As shown in figures 2(a)–(d) and 3(a)–(c), the predicted exponents are supported by computer simulations on Erdős–Rényi networks with neighbor degree $\kappa = 10$.

6.2. Regulatory

Our second example is gene regulatory dynamics governed, according to Michaelis–Menten (MM) model [14], by:

$$\frac{dx_i}{dt} = -Bx_i^a + \omega \sum_{j=1}^N A_{ij} \frac{x_j^h}{1+x_j^h}. \quad (24)$$

Under this framework, $M_0(x_i) = -Bx_i^a$, describing degradation ($a = 1$), dimerization ($a = 2$) or a more complex bio-chemical depletion process (fractional a), occurring at a rate B ; without loss of generality we set here $B = 1$. The activation interaction is captured by the Hill function of the form $M_1(x_i) = 1$, $M_2(x_j) = x_j^h/(1+x_j^h)$, a *switch-like* function that saturates to $M_2(x_j) \rightarrow 1$ for large x_j , representing node j 's positive, albeit bounded, contribution to node i activity, $x_i(t)$.



The appropriate case for a continuous transition is when $a = h$ to satisfy the relation mentioned above, $\Gamma_0 = \Pi_0$. Thus, $\Gamma_0 = a$ and Γ_1 does not exist, $\Pi_0 = a$ and $\Pi_1 = 2a$, $\Lambda_0 = 0$ and $m = 2a$. Hence, the equilibrium critical exponents, using equations (11)–(13), are $\beta = 1/a$, $\delta = 2a$ and $\gamma = 2 - 1/a$. The transient critical exponents, obtained by equations (20)–(22), are $\varphi = 1/(2a - 1)$, $\phi = 2 - 1/a$ and $\theta = 1 - 1/(2a)$. The simulation results compared to these predictions are presented in figures 2(e)–(h) and 3(d)–(f). An interesting result is for $a = 1/2$ or less when some of the exponents become zero or even negative. See figure 2(h) for $a = 1/2$ (circles) where, as predicted, the susceptibility does not diverge at criticality in contrast to all other examples. This indicates an essential change in the effect of the control. While for other cases the response at criticality to control is very high, in regulatory dynamics with $a = 1/2$, the system responds uniformly to control. See SI figure S3 for the faster decay at criticality for $a \leq 1/2$ compared to the power-law for $a > 1/2$.

6.3. Opinion

Our final example is a model (which we call BLSS) for opinion dynamics [18],

$$\frac{dx_i}{dt} = -x_i + \omega \sum_{j=1}^N A_{ij} \tanh(\alpha x_j). \quad (25)$$

The sign of x_i describes the agent i 's qualitative stance towards a binary issue of choice (e.g. the preference between two candidates). The absolute value of x_i quantifies the strength of this opinion, or the convincing

level. This model treats opinion dynamics as a purely collective, self-organized process without any intrinsic individual preferences. Hence, the opinions of agents without social interactions decay toward the neutral state 0, which is ruled by the self-dynamics function, $M_0(x_i) = -x_i$. The interaction ij is captured by $M_1(x_i) = 1$ and $M_2(x_j) = \tanh(\alpha x_j)$. This odd nonlinear shape guarantees that an agent j influences others in the direction of its own opinion's sign, with a level that increases monotonically with its convincing level, albeit with saturation since the social influence of extreme opinions is capped. We set here $\alpha = 1$. The leading orders of the dynamic functions are, therefore, $\Gamma_0 = 1$ and Γ_1 does not exist, $\Pi_0 = 1$ and $\Pi_1 = 3$, $\Lambda_0 = 0$ and $m = 3$. Hence, the obtained equilibrium exponents derived from equations (11)–(13), are $\beta = 1/2$, $\delta = 3$ and $\gamma = 1$. These are the same critical exponents as those of Ising model for ferromagnetism in mean-field [1]. This is not surprising since the model used to describe spin dynamics [43], is similar to equation (25) albeit not included in equation (1). The transient exponents, yielded from equations (20)–(22), are $\varphi = 1/2$, $\phi = 1$ and $\theta = 2/3$. These results are supported via computer simulations in figures 2(i)–(l) and 3(g)–(i).

7. Discussion

In this paper, we considered a general class of complex dynamical systems, analyzed by a general framework, to explore the behavior of the system near criticality of a PT. This PT captures the interplay between structure and dynamics such that weak connectivity yields a suppressed activity while strong connectivity leads to an active state of the system. Our main focus is on the impact of controlling dynamics of a fraction of nodes on the system state close to criticality. By only leading terms of the dynamic functions, we construct both *equilibrium* critical exponents and *transient* critical exponents. Applying our framework to three examples we reveal distinct universality classes, indicating the essential different effects of external control in these systems.

Yet, several directions are still needed to be explored for going beyond our results. Here we assumed that the controlled nodes are selected randomly. However, external control is likely non-random but could be rather targeted. For instance, a localized control, i.e. a source node and its neighbors and next neighbors and so on, might exhibit a considerably unequal effect. Another reasonable targeted control is by selecting the high-degree nodes. This case becomes more interesting for scale-free (SF) networks which show a diverse degree distribution. Speaking about SF networks, in our analysis we used a mean-field approach, relying upon an assumption of relatively small fluctuations, which is challenged by SF networks. Thus, networks with a very broad degree distribution demand further analysis. An additional natural interesting extension for this work could be the analysis of more general form of dynamics which are not included in equation (1). This could include the ferromagnetic dynamics [43], which is the main example in the literature for an external field in continuous PT [1]. Finally, here we focused on continuous transitions, however, an extension of our analysis can explore in a similar way abrupt transitions, see [28, 44].

Data availability statement

No datasets were generated or analyzed during the current study.

Acknowledgments

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Author contributions

Both authors designed the research and wrote the article. H S performed the analytical derivations and the computer simulations.

Code availability

All codes to reproduce, examine and improve our proposed analysis are available at <https://github.com/hillel26/ControlDynamicsCriticalExponents.git>.

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