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The Combined Effect of Connectivity and Dependency Links on Percolation of Networks

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Abstract Percolation theory is extensively studied in statistical physics and mathematics with applications in diverse fields. However, the research is focused on systems with only one type of links, connectivity links. We review a recently developed mathematical framework for analyzing percolation properties of realistic scenarios of networks having links of two types, connectivity and dependency links. This formalism was applied to study Erdős-Rényi (ER) networks that include also dependency links. For an ER network with average degree $\bar{k}$ that is composed of dependency clusters of size s, the fraction of nodes that belong to the giant component, P_∞, is given by $P_\infty = p^{s-1}[1 - \exp(-\bar{k} p P_\infty)]^s$ where $1 - p$ is the initial fraction of randomly removed nodes. Here, we apply the formalism to the study of random-regular (RR) networks and find a formula for the size of the giant component in the percolation process: $P_\infty = p^{s-1}(1 - r^k)^s$ where r is the solution of $r = p^s(r^{k-1} - 1)(1 - r^k) + 1$, and k is the degree of the nodes. These general results coincide, for $s = 1$, with the known equations for percolation in ER and RR networks respectively without dependency links. In contrast to $s = 1$, where the percolation transition is second order, for $s > 1$ it is of first order. Comparing the percolation behavior of ER and RR networks we find a remarkable difference regarding their resilience. We show, analytically and numerically, that in ER networks with low connectivity degree or large dependency clusters, removal of even a finite number (zero fraction) of the infinite network nodes will trigger a cascade of failures that fragments the whole network. Specifically, for any given s there exists a critical degree value, $\bar{k}_{\min}$, such that an ER network with $\bar{k} \leq \bar{k}_{\min}$ is unstable and collapse when removing even a single node. This result is in contrast to RR networks where such cascades and full fragmentation can be triggered only by removal of a finite fraction of nodes in the network.

Keywords Percolation · Networks · Cascade of failures · Dependency links

In honor of Professor Cyril Domb, our teacher.

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1 Introduction

Percolation is a primary model for disordered systems which is extensively studied both in mathematics [1–3] and in statistical physics [4–7] and have applications in diverse phenomena in different disciplines ranging from porous materials [8] and branched polymers [9] to forest fires [10] and epidemics spreading [11]. In network science [12–25], percolation theory is particular useful in studying robustness of networks to a failure of their components [26–29]. For a given network, when nodes are removed with probability $1-p$, or occupied with probability p, the percolation theory explains the existence of a critical probability, p_c, such that for p below p_c the network is composed of isolated small clusters but for p above p_c a giant cluster, of fraction P_∞ of the network, spans the entire network. The percolation process represents a phase transition of second-order where P_∞ (the order parameter) approaches continuously to zero when p approaches p_c from above. The value of p_c is considered as a measure for the network robustness. When p_c is smaller, the network is more robust since more nodes have to be removed from the network in order to fragment it completely. The results of percolation theory were useful, for example, to evaluate the resilience of the Internet against random [26] or malicious attacks [27–31] or to develop efficient immunization strategies [31–34].

Percolation theory assumes that the network elements are connected by one type of links, connectivity links. However, in real systems, usually there exist two types of links, connectivity links for the function of the network and dependency links which represent that the function of given elements depend crucially on others. For example, a dependency link between node i and node j means that if node i fails node j also fails, and vice versa. Network models containing both connectivity and dependency links were introduced recently for two interdependent networks [35, 36] and for single networks [37, 38]. For further recent studies of interdependent networks see Refs. [39–42].

Here, we review the general formalism for analyzing a single network composed of connectivity and dependency links and the results obtained for Erdős-Rényi (ER) type connectivity networks with dependency clusters. We apply here the formalism to analyze random-regular (RR) connectivity networks with dependency clusters. We also compare between the results of ER and RR networks and show that in the presence of dependency links, RR networks are significantly more robust compared to ER networks. This is in contrast to the case of no dependency links where ER are more robust than RR networks.

2 General Formalism

When nodes fail in a network containing both connectivity links and dependency clusters, two different processes occur. (i) Connectivity links connected to the failed nodes also fail, causing other nodes to disconnect from the network (connectivity step). (ii) A failing node cause the failure of all the other nodes of its dependency cluster, even though they are still connected via connectivity links (dependency step). Thus, a node that fails in the connectivity step leads to the failure of its entire dependency cluster, which in turn leads to a new connectivity step, which further leads to a dependency step and so on. Once the cascade process is triggered it will stop only if nodes that fail in one step do not cause additional failure in the next step.

In order to describe the network size during the cascade and, in particular, to find its size at steady state, two functions are defined, $g_D(t)$ and $g_p(t)$, evaluating respectively the effect of the dependency clusters and the effect of the connectivity process on the network at each step of the cascade. After random removal of $1-t$ of the nodes, the size of the giant component is given by $g_p(t)$ and, equivalently, the part of the network that is not dependent on the removed nodes (and, thus, remain functional) is given by $g_D(t)$. Thus, when applying the dependency process on a network of size x the remaining functional nodes consisting of a fraction $g_D(x)$, which is a fraction $\phi = xg_D(x)$ of the N nodes in the original network. Similarly, applying the connectivity process on a network of size y results in a remaining giant component consisting of a fraction $g_p(y)$ which is a fraction $\phi = yg_p(y)$ of the N nodes in the original network.

The state of the network at steady state is given, according to [38], by the two equations: $y = pg_D(x)$ and $x = pg_p(y)$, which can be reduced to a single equation:

$$x = pg_p(pg_D(x)). \tag{1}$$

Solving (1) we obtain the size of the network at the end of a cascade initiated after random removal of $1-p$ of the nodes.

This formalism is general for any network having connectivity and dependency links. In Sect. 3 we evaluate explicitly the function g_p, describing the connectivity process and in Sect. 4 we evaluate the function g_D, describing the action of the dependency links.

3 Connectivity Process

The percolation process can be solved analytically using the apparatus of generating functions. We introduce the generating function of the degree distribution $G_0(\xi) = \sum_k P(k)\xi^k$, where $P(k)$ is the probability of a node to have k connectivity links [43–45]. Analogously, we introduce the generating function of the underlining branching processes, $G_1(\xi) = G_0'(\xi)/G_0'(1)$. Random removal of a fraction $1-t$ of nodes will change the degree distribution of the remaining nodes, so the generating functions of the new distribution are equal to the generating functions of the original distribution with the argument equal to $1-t(1-\xi)$ [43]. Thus, the fraction of nodes that belong to the giant component after the removal of $1-t$ nodes is [44, 45]:

$$g_p(t) = 1 - G_0[1 - t(1-u)], \tag{2}$$

where $u = u(t)$ satisfies the self-consistency relation

$$u = G_1[1 - t(1-u)]. \tag{3}$$

Equations (2) and (3) describe generally random networks having any degree distribution. Specifically, we analyze two random network models that can be solved explicitly. In 3.1 we consider ER networks, whose connectivity degrees are Poisson-distributed [46–48], and in 3.2 we consider RR networks, where all nodes have the same connectivity degrees k (see Fig. 1).

3.1 Erdős-Rényi (ER) Networks

Erdős-Rényi (ER) networks are networks with a Poissonian degree distribution,

$$P(k') = \frac{k^{k'}}{k'!}e^{-\bar{k}}, \tag{4}$$

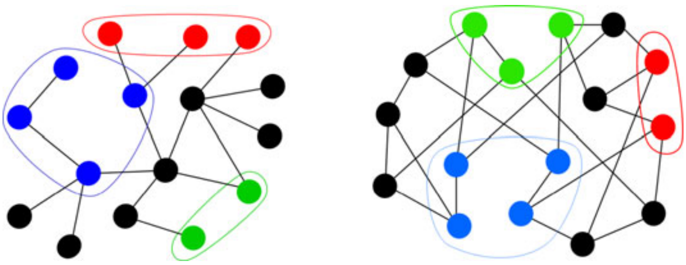


Fig. 1 (Color online) Connectivity networks with dependency clusters. The edges represent connectivity relations, while the (blue, red and green) groups surrounded by curves represent dependency relations between all the nodes of the same group (color). The dependency relations can be between very "*far*" nodes in the connectivity network. (**a**) ER network with dependency clusters. (**b**) RR network with dependency clusters where all nodes have same degree $k = 3$

where k' is the degree of the node and $\bar{k}$ is the mean degree. For ER network with average degree $\bar{k}$ the generating functions are [49],

$$G_0(\xi) = G_1(\xi) = \exp[\bar{k}(\xi - 1)]. \tag{5}$$

Thus, according to (2) and (3),

$$g_p(t) = 1 - u \tag{6}$$

where u satisfies the self consistent equation

$$u = e^{-\bar{k}t(1-u)}. \tag{7}$$

The steady state at the end of the cascade, given in (1), becomes

$$\begin{aligned} x &= p(1-u) \\ u &= e^{-\bar{k}pg_D(x)(1-u)}, \end{aligned} \tag{8}$$

which can be reduced to a single equation for u

$$u = e^{-\bar{k}pg_D(p(1-u))(1-u)}. \tag{9}$$

In order to present u, obtained from (9), in terms of P_∞ recall that the size of the giant component of the network at the steady state, ϕ_∞, is given by $\phi_\infty = xg_D(x)$, where x is the solution of (8). The remaining fraction of the network at the end of the cascade, P_∞, out of the initial fraction p is given by $P_\infty \equiv \phi_\infty / p$. Thus, we get the relation $P_\infty p = xg_D(x)$. Finally, using (8) a simple equation for P_∞ is obtained

$$P_\infty = -\frac{\ln u}{\bar{k}p}, \tag{10}$$

where u is the solution of (9).

3.2 Random-Regular (RR) Networks

In RR networks all nodes have the same connectivity degree k. In this case the generating function of the degree distribution is $G_0(\xi) = \xi^k$ and the generating function of the underlying branching process is $G_1(\xi) = \xi^{k-1}$, thus, $G_0(\xi) = [G_1(\xi)]^{\frac{k}{k-1}}$. According to (2) and (3)

$$g_p(t) = 1 - u^{\frac{k}{k-1}}, \tag{11}$$

where u satisfies the self consistent equation

$$u = [1 + (u-1)t]^{k-1}. \tag{12}$$

The steady state at the end of the cascading process is given by (1)

$$\begin{aligned} x &= p(1 - u^{\frac{k}{k-1}}) \\ u &= [1 + (u-1)pg_D(x)]^{k-1}. \end{aligned} \tag{13}$$

Once system (13) is solved, the size of the giant component is given by $P_\infty = xg_D(x)/p$.

4 Dependency Process

In the general case, when dependency links exist, each node belongs to a dependency group of size s with a probability $q(s)$ so that the number of groups of size s is equal to $q(s)N/s$. Since after random removal of $1-t$ of the nodes each group of size s remains functional with a probability t^s, the total number of nodes that remain functional is given by $\sum_{s=1}^{\infty} q(s)Nt^s$. Thus, we define the function $g_D(t)$ as the fraction of nodes that remain functional out of the tN nodes that were not removed,

$$g_D(t) \equiv \sum_{s=1}^{\infty} q(s)t^{s-1}. \tag{14}$$

In particular, when all dependency clusters have the same size, s,

$$g_D(t) = t^{s-1}. \tag{15}$$

5 ER and RR Networks with Dependency Clusters of the Same Size

5.1 ER Network with Dependency Clusters of the Same Size

The percolation of an ER network with average connectivity degree k and dependency clusters of size s is described by substituting (15) into (9)

$$u = e^{-\bar{k}p^s(1-u)^s}, \tag{16}$$

where $1-p$ is the initial fraction of removed nodes. The size of the giant cluster, P_∞, is given, using (16) and (10), by [38],

$$P_\infty = p^{s-1}[1 - \exp(-\bar{k}pP_\infty)]^s \quad (ER). \tag{17}$$

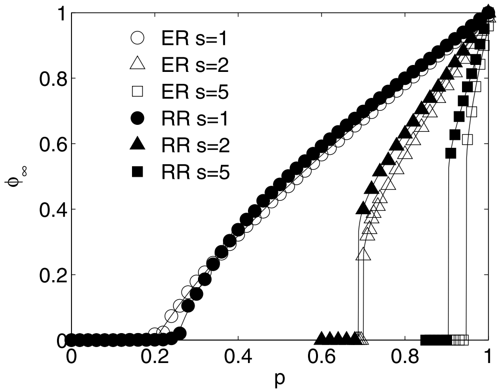


Fig. 2 Simulation results of the relative size of the giant component, $\phi_\infty \equiv P_\infty p$, versus p for ER networks (*open symbols*) and RR networks (*full symbols*) with connectivity degree $\bar{k} = 5$ for ER and $k = 5$ for RR, compared with the theory (*solid lines*) as obtained from (17) for ER and from (24) for RR. For the case of $s = 1$ there are no dependency clusters and the regular percolation process leads to the known second-order phase transition. For $s \geq 2$, a first order phase transition characterize the percolation process represented by discontinuity of P_∞ at p_c. Both the regular second-order and the first-order percolation transitions obey (17) for ER and (24) for RR. Note that while for $s = 1$ ER networks are more stable than RR networks (p_c is smaller in ER than in RR) as s increases the RR networks become more stable compared to ER

Equation (17) coincides for $s = 1$ (a node depends only on itself, i.e., no dependency links) with the known Erdős-Rényi equation [46–48], $P_\infty = 1 - \exp(-\bar{k} p P_\infty)$, for a network without dependency relations. Moreover, for $s = 2$, (17) yields the result obtained in [37] for the case of dependency pairs. The cases of $q(s)$ being Gaussian or Poissonian were also studied by Bashan et al. [38].

Figure 2 shows the size of the giant cluster, $\phi_\infty \equiv P_\infty p$, versus the fraction of nodes, p, remaining after an initial random removal of $1 - p$, for the case of ER network with fixed size of dependency clusters s. The case of $s = 1$, where each node depends only on itself, is the known regular second order percolation transition. For any $s \geq 2$, a first order phase transition characterizes the percolation process. Both the regular and the new first order percolation obey (17).

Next, we describe how to find analytically the percolation threshold, p_c, for the case of ER network with a fixed size s of dependency clusters [38]. Equation (16), which is the condition for a steady state, have a trivial solution at $u = 1$, which corresponds, by (10), to a complete fragmentation of the network. For large p there is another solution of $0 < u < 1$, corresponding to the existence of a giant component being a finite fraction of the network. Therefore, the critical case corresponds to satisfying both the tangential condition for (16),

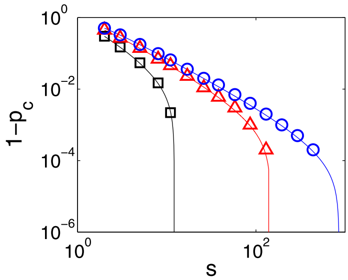
$$1 = \bar{k} p^s s u (1 - u)^{s-1}, \tag{18}$$

as well as (16). Thus, combining (18) and (16) we obtain a closed-form expression for the critical value, u_c,

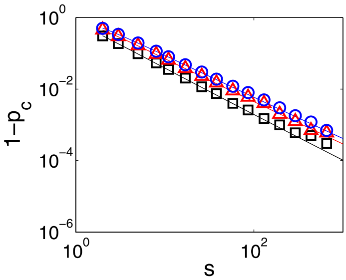
$$u_c = \exp\left(\frac{u_c - 1}{s u_c}\right). \tag{19}$$

Once u_c is found, we obtain p_c by substituting it into (18),

$$p_c = [\bar{k} s u_c (1 - u_c)^{s-1}]^{-1/s}. \tag{20}$$



(a) ER networks.



(b) RR networks.

Fig. 3 (Color online) Critical probability p_c for (**a**) ER and (**b**) RR networks with dependency clusters of size s, for different connectivity degree $\bar{k}$: (black) *squares* for $\bar{k}=5$, (red) *triangles* for $\bar{k}=8$ and (blue) *circles* for $\bar{k}=10$. While for ER [Fig. 3(**a**)] $1-p_c$ drastically drops to zero ($p_c=1$) at $s_{\max}$, in RR [Fig. 3(**b**)] $1-p_c$ scale as power law, see (28), and never reaches zero

For $s=1$ we obtain the known result $p_c=1/\bar{k}$ of Erdős-Rényi [46–48]. Substituting $s=2$ in (19) and (20) one obtains $p_c{}^2=1/[2\bar{k}u_c(1-u_c)]$, which coincides with the exact result found in [37].

When (20) yields a value which is equal or larger than one for p_c, removal of even a finite number of nodes from an infinite network (zero fraction) will trigger a cascade of failures that fragments the whole network. Moreover, for any given s there is a minimal degree value,

$$\bar{k}_{\min}(s)=[su_c(1-u_c)^{s-1}]^{-1}, \tag{21}$$

such that p_c for an ER network with $\bar{k}\le\bar{k}_{\min}$ is equal to one and the network is extremely unstable. Similarly, for any given connectivity degree $\bar{k}$ there exists $s_{\max}$, given by (20) under the condition of $p_c=1$, such that for $s>s_{\max}$, the value of p_c is larger than one (see Fig. 3(a)). In Fig. 3 we plot the values of p_c as a function of s for several $\bar{k}$ values.

5.2 RR Network with Dependency Clusters of the Same Size s

In the case of RR network with fixed size s of dependency clusters, (13) and (15) can be reduced to

$$u=[(u-1)p^s(1-u^{\frac{k}{k-1}})^{s-1}+1]^{k-1}. \tag{22}$$

Introducing a new variable $r\equiv u^{\frac{1}{k-1}}$, (22) becomes

$$r=p^s(r^{k-1}-1)(1-r^k)^{s-1}+1. \tag{23}$$

The size of the giant cluster, P_∞, after random removal of $1-p$ of the network nodes, is given by

$$P_\infty=p^{s-1}(1-r^k)^s \quad (RR). \tag{24}$$

Similar to the case of ER networks, as shown in Fig. 2, the percolation process of RR networks, given in (23, 24), represents a phase transition. For $p>p_c$ the size of the network at the end of the cascade process, P_∞, is finite, while for $p<p_c$ the network becomes completely fragmented. The transition point is given by the tangential condition of (23)

$$1=(1-r_c)r_c^{k-2}\left[\frac{(s-1)kr_c}{1-r_c^k}+\frac{k-1}{1-r_c^{k-1}}\right]. \tag{25}$$

Solving (25) we obtain the transition point, p_c,

$$p_c^s = \frac{1-r_c}{(1-r_c^{k-1})(1-r_c^k)^{s-1}}. \tag{26}$$

The size of the giant cluster, P_∞, at p_c is obtained by substituting r_c into (24).

Next we show analytically that for RR networks p_c is always smaller than 1. This is in contrast to ER networks where p_c can be equal or larger than 1, (see (21)). As shown in Fig. (3), as the size of the dependency clusters, s, increases p_c increases and the network becomes more vulnerable. When the connectivity degree, k, increases the system becomes more robust and p_c becomes smaller. In order to check if RR networks can become totally fragmented in removal of zero fraction of its nodes, namely, $p_c \geq 1$, we evaluate p_c in the limit of large s. For $s \gg 1$, r_c is evaluated from (25)

$$r_c|_{s\gg 1} = \left(\frac{1}{sk}\right)^{\frac{1}{k-1}}. \tag{27}$$

Substituting r_c into (26) and taking $s \gg 1$ we obtain

$$p_c|_{s\gg 1} \simeq \left[1-\left(1-\frac{1}{k}\right)\left(\frac{1}{sk}\right)^{\frac{1}{k-1}}\right]^{\frac{1}{s}} \simeq 1-(1-1/k)k^{\frac{-1}{k-1}}s^{\frac{-k}{k-1}}. \tag{28}$$

This formula can explain the power law behavior as a function of s and the slopes seen in Fig. 3(b). Since $0 < r_c < 1$ and $k > 1$, we obtained that the critical point of the percolation transition p_c of RR networks is always smaller than 1 for any given s.

6 Summary

The synergy between the connectivity and the dependency processes in networks having links of two types, connectivity and dependency links, leads to cascade of failures that change the percolation properties of the system. The new percolation laws, given by (17) and (24) for ER and RR networks respectively, predict for both ER and RR networks a second-order percolation transition where no dependency clusters are present and a first-order transition for networks with dependency clusters. However, the two topologies are dramatically different regarding their stability in the case of large dependency clusters. In ER networks with low connectivity degree or large dependency clusters, removal of even a finite number (zero fraction) of the network nodes may trigger a cascade of failures that fragments the whole network. This is in contrast to RR networks where such cascades can be triggered only by removal of a finite fraction of nodes in the network.

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