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PAPER

Cascading failures in anisotropic interdependent networks of spatial modular structures

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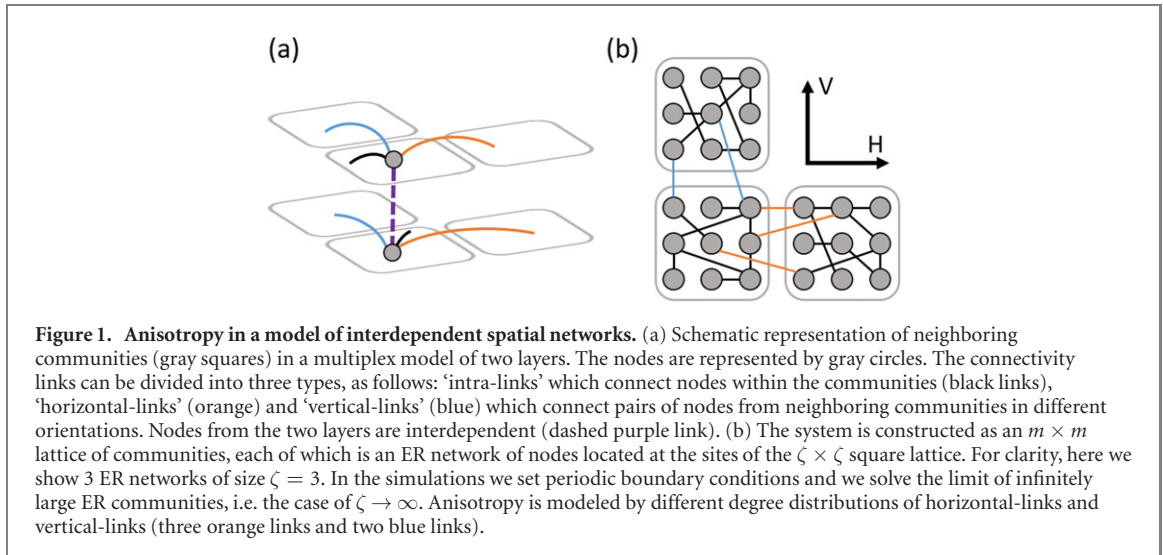
Abstract

The structure of real-world multilayer infrastructure systems usually exhibits anisotropy due to constraints of the embedding space. For example, geographical features like mountains, rivers and shores influence the architecture of critical infrastructure networks. Moreover, such spatial networks are often non-homogeneous but rather have a modular structure with dense connections within communities and sparse connections between neighboring communities. When the networks of the different layers are interdependent, local failures and attacks may propagate throughout the system. Here we study the robustness of spatial interdependent networks which are both anisotropic and heterogeneous. We also evaluate the effect of localized attacks having different geometrical shapes. We find that anisotropic networks are more robust against localized attacks and that anisotropic attacks, surprisingly, even on isotropic structures, are more effective than isotropic attacks.

1. Introduction

Many real-world systems are well correlated with a population density that is highly non-uniform and concentrates in large cities or is spread along seacoasts, rivers, or major transportation routes. Such systems are influenced by geographical and social features and usually combine both spaciality on large scales and randomness on small scales. More specifically, in many cases such as infrastructure networks, the connections within the cities are dense and almost uniform, while the connections between cities are mainly between nearby cities. These factors were the motivation behind the profound studies of spatial networks [1–12], either homogeneous or heterogeneous and composed of communities. However, the connections between the cities do not have to be isotropic. For example, if in one axis there are more topographic obstacles (e.g. mountains or rivers), then in that axis there will be fewer connections than in the other axes. Yet, most models of spatial networks are isotropic, i.e. have no preferred orientation in the network structure.

In addition, many realistic systems can be considered as strongly interdependent multi-layered networks. Among the notable examples are modern infrastructure networks [13–15], and social networks, in which individuals take part in multiple social networks [16–18]. Therefore, much consideration was given to the study of interdependent networks from various aspects [19, 20]. In particular, percolation theory was used to study the robustness of such networks [21–33] and to analyze cascading failures resulting from various attacks [34–40]. One of the key results in these types of studies is that often localized attacks are significantly more effective when compared to random attacks. To the best of our knowledge, in studies of localized attacks, the attack itself has always been uniform within a certain radius [41–44] although in the real world this ‘uniformity condition’ usually does not occur. For example, natural disasters (such as an



earthquake) and malicious targeted attacks can be anisotropic and even contained within a single axis. Here, we take into consideration **anisotropy** in modeling both in the systems and in the localized attacks.

2. Model

Here we introduce an anisotropic model of interdependent spatially embedded networks with communities. The model assumes that the entire 2D territory is divided into squared communities of size $\zeta \times \zeta$ representing cities or densely populated areas, see figure 1. For simplicity, we assume a multiplex system with two topologically distinct network layers that share the same set of nodes. The interdependence between the layers is introduced as follows: if a node becomes dysfunctional in one layer it is also dysfunctional in the other layer. In each layer, the connectivity links within a community (‘intra-links’) are chosen from a given degree distribution. In this manuscript, we focus on the case in which this distribution forms an ER network (i.e. the links are connected at random). This choice is motivated by both the observation that in many realistic systems (e.g. infrastructure networks in a country), the community itself (like transportation in a city in which it is easy to get from one place of the city to another) is approximately homogeneous and exhibits no spatial structure, as well as from the fact that it allows simplifying the analytical equations which enable to get deep insight and focusing on the large-scale spatial structure. The links connecting nodes in two different communities (‘inter-links’) can only connect neighboring squares in each network, horizontally or vertically, as illustrated in figure 1(b). Each node has a degree k_{intra} of intra-links, a degree k_H of horizontal-links, a degree k_V of vertical-links, and the total degree is $k_{\text{total}} = k_{\text{intra}} + k_H + k_V$. We assume that k_{intra} , k_H and k_V are independent random variables taken from three different degree distributions which are characterized by average degrees $\langle k_{\text{intra}} \rangle$, $\langle k_H \rangle$ and $\langle k_V \rangle$, respectively. The anisotropy of the system is specified by the parameter

$$\gamma = \frac{\langle k_H \rangle}{\langle k_H \rangle + \langle k_V \rangle}, \quad (1)$$

which is the ratio between the degree of the horizontal-links and the inter-links (and thus $\gamma = 1/2$ is the isotropic case), and the heterogeneity of the system is specified by the ratio between the degree of the inter-link and the total degree $\alpha = (\langle k_H \rangle + \langle k_V \rangle) / \langle k_{\text{total}} \rangle$.

Here we study the robustness of spatial anisotropic multiplex networks to different forms of localized attacks. Initial damage in a multiplex network spreads in a process in which failures of some nodes lead to failures of other nodes and so on. In the cascading failures process, a node fails if it is no longer connected to the giant component in one of the layers. In our simulations, we perform the cascading failures after an initial attack as follows: from the set of the remaining nodes we remove all nodes that are not in the giant component of the first layer, and then from the remaining nodes we remove all nodes that are not in the giant component of the second layer. These two steps are repeated back and forth until there are no nodes to remove, i.e. when all the remaining nodes are part of the mutual giant component (MGC). We consider a network as functioning if the MGC is of size $O(N)$, where N is the number of nodes in the network. Accordingly, in our spatial multilayer network model, the network is considered to be resistant to

a localized attack if the damage spreads to a finite number of communities and does not reach the edges of the system.

3. Analytical approach

In our recent study [44], we developed a mathematical framework that consists of general equations for the MGC size of a multiplex model of $m \times m$ ER communities connected as a lattice. However, in that study, we applied the equations only for the case of an isotropic network and we analyzed only the effect of isotropic localized attacks on the functionality of the network. Here, we study a more general and realistic case, of an anisotropic model with different average degrees for the vertical links $\langle k_V \rangle$ and the horizontal links $\langle k_H \rangle$, as well as anisotropic localized attacks. For a network with given levels of heterogeneity and anisotropy, we determine whether it remains functional after the cascading failure induced by isotropic or anisotropic initial damages. The initial damage in community i (for i from 1 to m^2) is expressed by the parameter p_i , which is defined as the fraction of nodes that survived as a result of the damage. For example, in the case of initial damage which is precisely removing all nodes in community j , we set $p_j = 0$ and $p_i = 1$ for $i \neq j$. Throughout the manuscript, we refer to this case as ‘removing a community’. Here the removed community is always chosen at random and due to the periodical boundary conditions, it can be considered, for convenience, as the ‘center’ of the network (this is true for all other localized attacks as well).

In the analytical model we consider a multiplex network in which the layers are generated interdependently but with the same degree distributions (and in particular with the same average degree), and each layer is composed of a lattice of infinitely large ER communities ($\zeta \rightarrow \infty$) that are characterized by Poisson degree distribution. Using the analytical formalism from our recent study [44] (for a more detailed discussion see appendices A and B), we obtain the following equation for the MGC size of each community i , $P_{\infty i}$,

$$P_{\infty i} = p_i \cdot \left[1 - e^{\sum_j \langle k_{ij} \rangle P_{\infty j}} \right]^2, \quad (2)$$

where

$$\langle k_{ij} \rangle = \begin{cases} \langle k_{\text{intra}} \rangle & \text{for } i = j \\ \langle k_H \rangle / 2 & \text{if } j \text{ is horizontal neighbor of } i \\ \langle k_V \rangle / 2 & \text{if } j \text{ is vertical neighbor of } i \\ 0 & \text{else.} \end{cases} \quad (3)$$

Next, we use this analytical formalism (equation (2)) to evaluate the robustness of interdependent spatial systems characterized by heterogeneity and anisotropy by calculating the steady state of each community. In the appendix, we verify these analytical solutions through simulations.

4. Results

Figure 2 shows that for a given $\langle k_{\text{total}} \rangle$, the robustness of the network to a removal of one community (i.e. removing all nodes inside that community) is highly dependent on both parameters α and γ representing the heterogeneity and anisotropy, respectively. In particular, in the phase diagrams for $\langle k_{\text{total}} \rangle = 2.48, 2.49, 2.5$ (figures 2(a)–(c)), the steady state of the system can be in one of two extreme states: stable (in yellow)—in which the system remains functional, and unstable (in blue)—in which the entire system collapses. It is important to note that the regions of the stable and unstable states do not depend on the system size m (see also figure 7 in the appendix). In addition, the transition from stable to unstable is non-monotonic with the anisotropy of the network. For instance, for $\langle k_{\text{total}} \rangle = 2.5$ and $\alpha = 0.5$ (see figure 2(c)), the unstable region is only in a specific narrow region of γ that is located between the extreme cases of $\gamma = 0.5$ (isotropic network) and $\gamma = 0$ or $\gamma = 1$ (highly anisotropic network).

The blue and orange curves added to the phase diagrams (figures 2(a)–(c)) show that the unstable region is contained within a specific area which is between the two curves and to the right of their intersection. This area is defined by the following constraints: $\langle k_{\text{total}} \rangle - \langle k_H \rangle / 2 < k_c$ and $\langle k_{\text{total}} \rangle - \langle k_V \rangle / 2 < k_c$, where $k_c \approx 2.4554$ is the critical average degree below which a single ER multiplex collapses without any initial damage [22]. These constraints describe a case when removing a community causes its neighboring communities to reduce their average total degree below k_c . In this mentioned area, there is also a stable region because the communities are not isolated and therefore a community with a total degree smaller than k_c can be sustained by the communities next to it. In figure 2(d) we show the behavior of cascading failures in this stable region. We show that the initial damage of removing one community spreads further along the direction of the *higher* inter-degree.

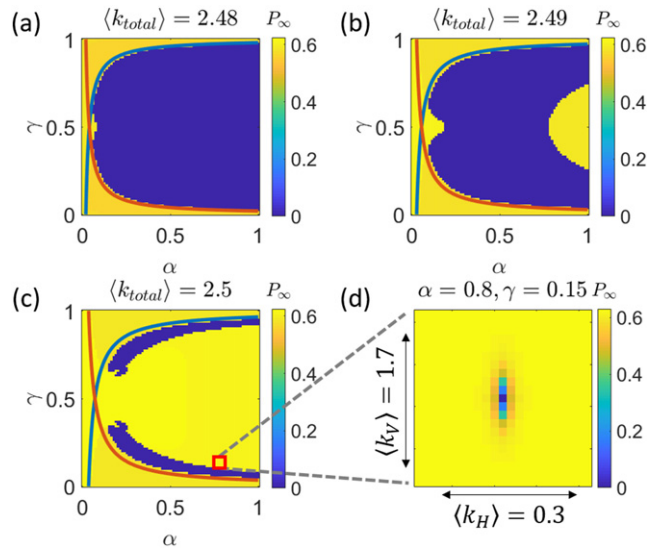


Figure 2. Robustness after removing one community. (a)–(c) Phase diagrams of the size of the functioning component, P_∞ , at steady state after initial removal of one community from a system of 441 communities (i.e. $m = 21$). The color of each pixel represents the relative size of the system's functioning component. In all phase diagrams we show the analytic solutions for $\langle k_{total} \rangle - \langle k_V \rangle / 2 = k_c$ (blue curve) and $\langle k_{total} \rangle - \langle k_H \rangle / 2 = k_c$ (orange curve), where $k_c = 2.4554$ [22]. (d) The network at steady state for the case of $\alpha = 0.8, \gamma = 0.15$ and $\langle k_{total} \rangle = 2.5$. Each pixel represents a community, and the color of the pixel represents the relative size of the functioning component of the community. Since $\gamma < 1/2$, the network is anisotropic with more vertical-links than horizontal-links (see equation (1)). The initial damage of removing one community causes cascading failures which spread more in the vertical axis than in the horizontal axis, i.e. along the direction of the higher degree.

Next, we determine the robustness of the system to various forms of isotropic and anisotropic localized attacks. We find that our system is metastable for a broad range of parameters, meaning that for a localized attack of a given shape there is a critical size above which the induced cascade of failures propagates through the whole system leading to its collapse. We observe that the size of the critical attack, which does not depend on the number of communities m , is significantly different for various shapes of attacks.

In figure 3, we analyze the case of isotropic localized attacks in a form of a circle with a radius r_h . We show that for a given $\langle k_{total} \rangle$ and γ , there is a critical α_c which has the following properties. For heterogeneity of $\alpha < \alpha_c$ the critical size r_h^c is of size $\approx m/2$ (system size), and for $\alpha > \alpha_c$, r_h^c does not depend on the system size and may contain more than one community. In figure 4(a), we present the critical attack size for attacks in a form of a strip, i.e. removing communities connected in a row. Indeed, the direction of the attack with respect to the direction of the higher degree (parallel or perpendicular) has a strong effect on the overall damage. We find that the attack is more efficient when the removed strip is orthogonal to the direction of the higher degree so it destroys more inter-links per one removed community. However, it is important to note that the number of initially damaged links is not the only factor in the overall effectiveness of the attack. This can be seen from the results in figure 2, where it is demonstrated that for a removal of a single community the attack effectiveness greatly differs depending on the level of anisotropy even though in both scenarios the same number of links are removed. In addition, we find that increasing the anisotropy of the network causes the critical length of the stripe attack to decrease. Note that we have chosen to focus on the range of anisotropy $0.3 < \gamma < 0.7$ because outside this range the system is always stable, i.e. the critical attack size is of the order of the system size (see figure 3). Note also that the small fluctuations are caused by the fact that the network is not continuous but consists of discrete communities, i.e. there are jumps when the damage crosses neighboring communities.

In addition, in figure 4(b), we analyze and demonstrate the evolution of cascading failures after a strip attack whose length is above the critical size. For these simulations, we solved the general iterative equations from [44], which represent the stages of the cascading failures, for our specific anisotropic model (see equation (7) in the appendix). We find that the damage propagates first in the axis with the higher inter degree and only after the damage reaches a certain distance in this axis, it begins to propagate also in the axis with the lower inter degree. From this, we conclude that the cascading along an axis is affected more by the size of the attack in the direction perpendicular to this axis than its size in the parallel direction.

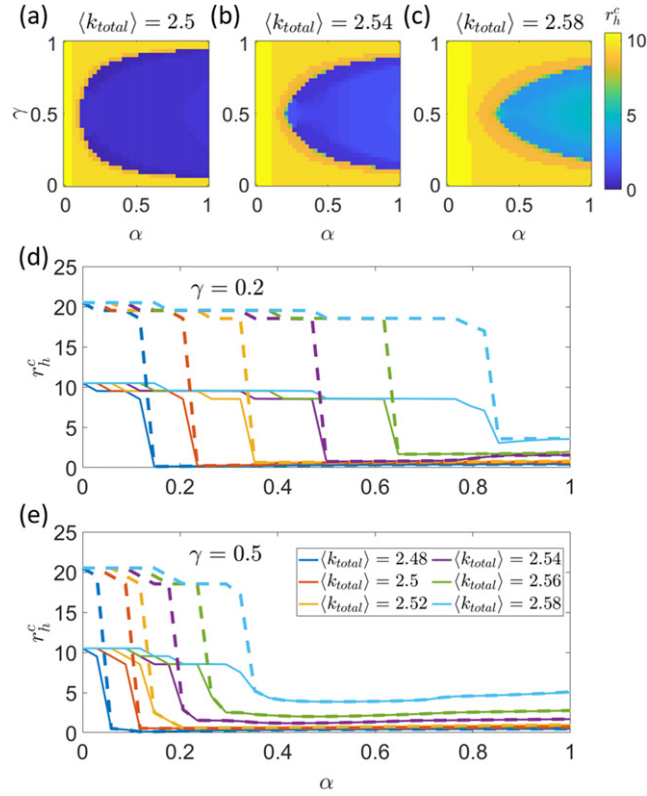


Figure 3. The critical radius as a function of α and γ . (a)–(c) Phase diagrams for the critical circular hole radius (in units of a community size), r_h^c , for different average total degree $\langle k_{total} \rangle$. In these simulations we set the system size to be $m = 21$. (d) and (e) The critical radius, r_h^c , for two values of m . The dashed lines show the r_h^c for the case of $m = 41$ and the continuous lines are for the case of $m = 21$. We find a metastable region, where a finite-size localized attack larger than r_h^c , which *does not* depend on the system size m , causes cascading failures and system collapse. The metastable region reduces as $\langle k_{total} \rangle$ increases, and for a fixed $\langle k_{total} \rangle$, there exists a critical value of α above which the damage of finite radius causes the collapse of the network. The critical α increases both by $\langle k_{total} \rangle$ and anisotropy.

Finally, we analyze the robustness of our anisotropic model with respect to localized attacks in a form of an ellipse with different flattening f_e , defined as $f_e = (a - b)/a$ where a and b are the semi-major and semi-minor axes, respectively. The case $f_e = 0$ corresponds to our earlier circular attack while $f_e \rightarrow 1$ represents a strip-like attack. Thus, the general ellipse case covers a wide range of anisotropic attacks between a strip attack and a circle attack. In figure 5, we show that the ellipse attack is more efficient for ellipses with higher flattening f_e , and that a strip attack is more efficient than any ellipse attack. We therefore conclude that for a given area of the attack, it is more efficient that the shape of the attack will be as anisotropic as possible.

5. Discussion

Many spatial networks are influenced by geographical features that can lead to an anisotropic structure in which the amount of links differs in different directions. In this work, we have introduced a new realistic model, which considers, for the first time, the aspect of network anisotropy. Using tools from percolation theory, we systematically study anisotropic systems and analyze their robustness to various isotropic and anisotropic localized attacks. We determine how the robustness of our network model depends on its heterogeneity and anisotropy. We also show that anisotropic attacks reveal significantly increased vulnerability compared to the considered earlier, simple circle-shaped attacks. Specifically, we find that a localized attack causes larger damage if the area of the attack is more elongated in the direction of the smaller inter degree, i.e. when the attack cuts more links that are in the direction of the higher inter degree. We find that even in isotropic networks the anisotropic localized attacks are more efficient than isotropic attacks, see the case $\gamma = 0.5$ in figure 5. In our future work, we will determine how the velocity of the cascading failures in the different axes depends on the parameters of the anisotropic network, and will try to answer the question of whether this velocity exhibits a scaling behavior.

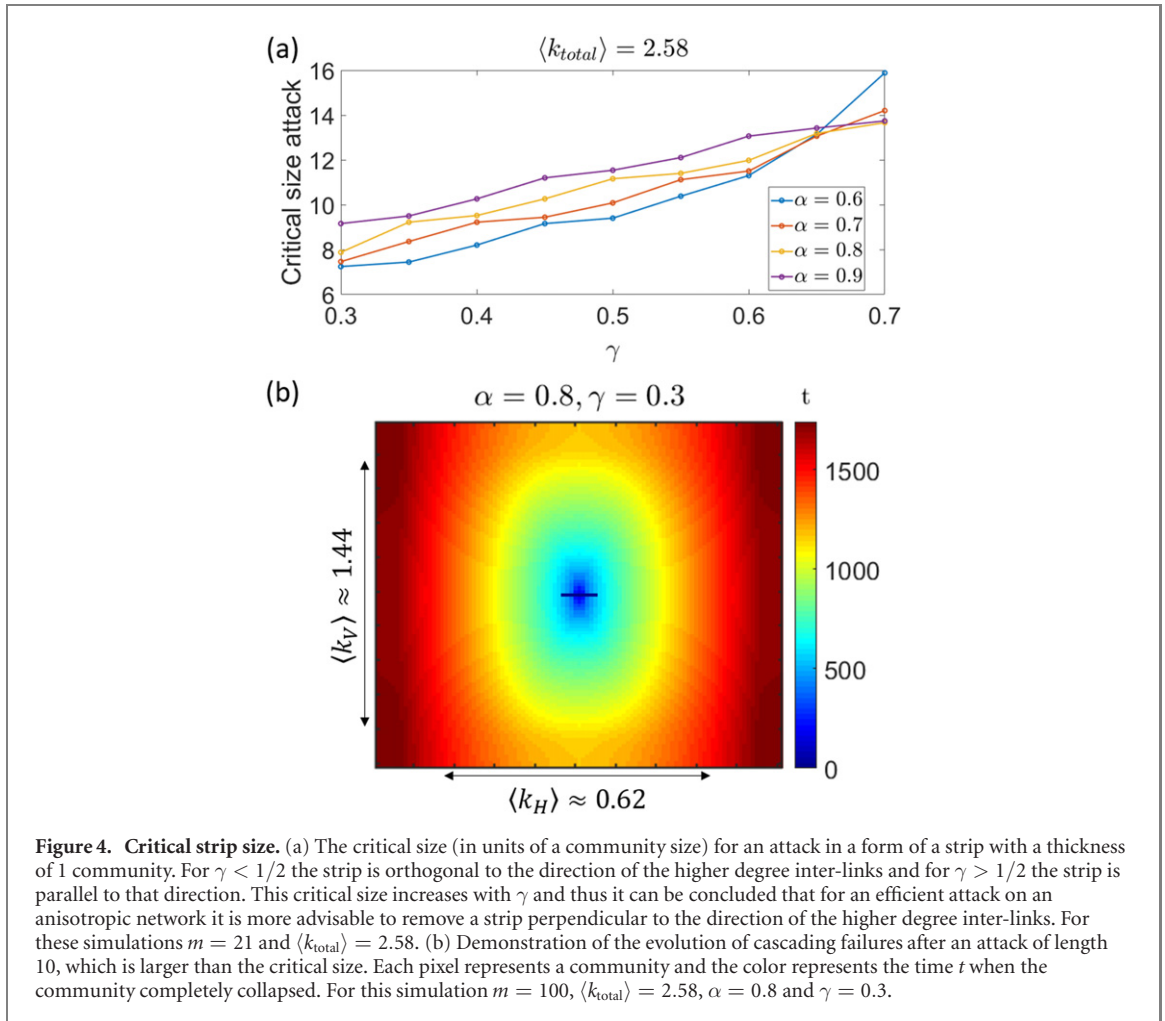


Figure 4. Critical strip size. (a) The critical size (in units of a community size) for an attack in a form of a strip with a thickness of 1 community. For $\gamma < 1/2$ the strip is orthogonal to the direction of the higher degree inter-links and for $\gamma > 1/2$ the strip is parallel to that direction. This critical size increases with γ and thus it can be concluded that for an efficient attack on an anisotropic network it is more advisable to remove a strip perpendicular to the direction of the higher degree inter-links. For these simulations $m = 21$ and $\langle k_{total} \rangle = 2.58$. (b) Demonstration of the evolution of cascading failures after an attack of length 10, which is larger than the critical size. Each pixel represents a community and the color represents the time t when the community completely collapsed. For this simulation $m = 100$, $\langle k_{total} \rangle = 2.58$, $\alpha = 0.8$ and $\gamma = 0.3$.

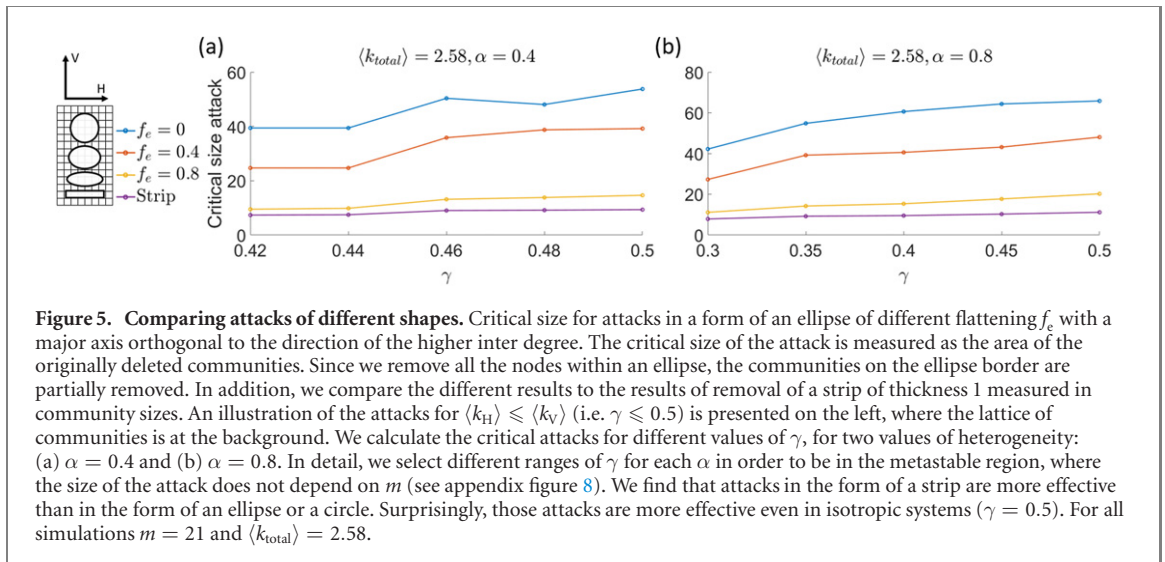


Figure 5. Comparing attacks of different shapes. Critical size for attacks in a form of an ellipse of different flattening f_e with a major axis orthogonal to the direction of the higher inter degree. Within an ellipse, the attack is measured as the area of the originally deleted communities. Since we remove all the nodes within an ellipse, the communities on the ellipse are partially removed. In addition, we compare the different results to the results of removal of a strip of thickness 1 measured in community sizes. An illustration of the attacks for $\langle k_H \rangle \leq \langle k_V \rangle$ (i.e. $\gamma \leq 0.5$) is presented on the left, where the lattice of communities is at the background. We calculate the critical attacks for different values of γ , for two values of heterogeneity: (a) $\alpha = 0.4$ and (b) $\alpha = 0.8$. In detail, we select different ranges of γ for each α in order to be in the metastable region, where the size of the attack does not depend on m (see appendix figure 8). We find that attacks in the form of a strip are more effective than in the form of an ellipse or a circle. Surprisingly, those attacks are more effective even in isotropic systems ($\gamma = 0.5$). For all simulations $m = 21$ and $\langle k_{total} \rangle = 2.58$.

Acknowledgments

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. The cascading failures equations

Here we present, for the reader's convenience, the detailed description of the general model and the general cascading failures equations based on our previous article [44], which are generalized here for the anisotropic model of this manuscript. We consider a multiplex of two interdependent networks A and B with an equal number of nodes, N , such that each node a_α in network A mutually depends on one and only one node b_α in network B and vice versa. Thus the pair of nodes (a_α, b_α) can be regarded as a single node $\alpha = 1, 2, \dots, N$ of a multiplex consisting of two different layers of links corresponding to networks A and B , so that if a_α fails b_α fails and vice versa. The multiplex is subdivided into M non-overlapping communities connected by a sparse symmetric adjacency matrix $C_{ij} = 0, 1$. In particular, C_{ij} may represent bonds in a square lattice, where each node is a community. The links in layers A and B may exist only if they connect nodes belonging to communities i and j for which $C_{ij} \neq 0$. Note that $C_{ii} = 1$, so that the nodes in any community may be connected to the nodes in the same community. Each community i is characterized by a set of its neighboring communities Ω_i , such that $C_{ij} = 1$ if and only if $j \in \Omega_i$. The number of communities in Ω_i is $K_i + 1$, where K_i is the degree of community i (in the square lattice case: $K_i = 4$). Any node α in community i is randomly connected to $k_{\alpha,ij}^A$ nodes in layer A and to $k_{\alpha,ij}^B$ nodes in layer B where $j \in \Omega_i$. If $j = i$ it is an intra-link and if $j \neq i$ it is an inter-link. We assume that partial degrees $k_{\alpha,ij}^A$ and $k_{\alpha,ij}^B$ are selected from arbitrary degree distributions P_{ij}^A and P_{ij}^B , respectively. If we also assume that the number of nodes in each community $N_i \rightarrow \infty$, then the problem can be solved using the apparatus of generating functions $G_{ij}^A(x) = \sum_{k=0} P_{ij}^A(k) x^k$ and $H_{ij}^A(x) = \sum_{k=0} (k+1) P_{ij}^A(k+1) x^k / \langle k_{ij}^A \rangle$, where $\langle k_{ij}^A \rangle = \sum_{k=1} k P_{ij}^A(k)$ is the average partial degree in layer A . The generating functions for layer B are the same except index A is changed to B .

The process of cascading failures is started by removing a fraction of $1 - p_i$ of nodes in each community independently of their partial degrees. As in reference [22], we assume that a node α survives if it belongs to the giant components of survived nodes in both layers A and B , i.e. to the MGC of the multiplex. Other conditions can be applied [45, 46]. As the cascade of failures progress, $p_i(t)$ denotes the number of survived nodes at stage t of the cascade, where $p_i(0) \equiv p_i$. Then, using the arguments of [22] we can find the iterative equations for the fraction of survived nodes $p_i(t)$ if we introduce functions $f_{ij}^X(t)$ with superscript $X = A, B$ which are the probabilities that the link from a node in community i to a node in community j in layer X does not lead to the giant component of survived nodes in layer X at the stage t of the cascade. Finally, we introduce vectors $\vec{p}(t) \in \mathbb{R}^M$ with components $p_i(t)$, $\vec{f}_X(t) \in \mathbb{R}^M$ with components $f_{ij}^X(t)$, and vector functions $\vec{\Phi}_X(\vec{f}, \vec{p}) \in \mathbb{R}^{[M \times M]}$ and $\vec{\Psi}_X(\vec{f}, \vec{p}) \in \mathbb{R}^{[M \times M]}$ with components:

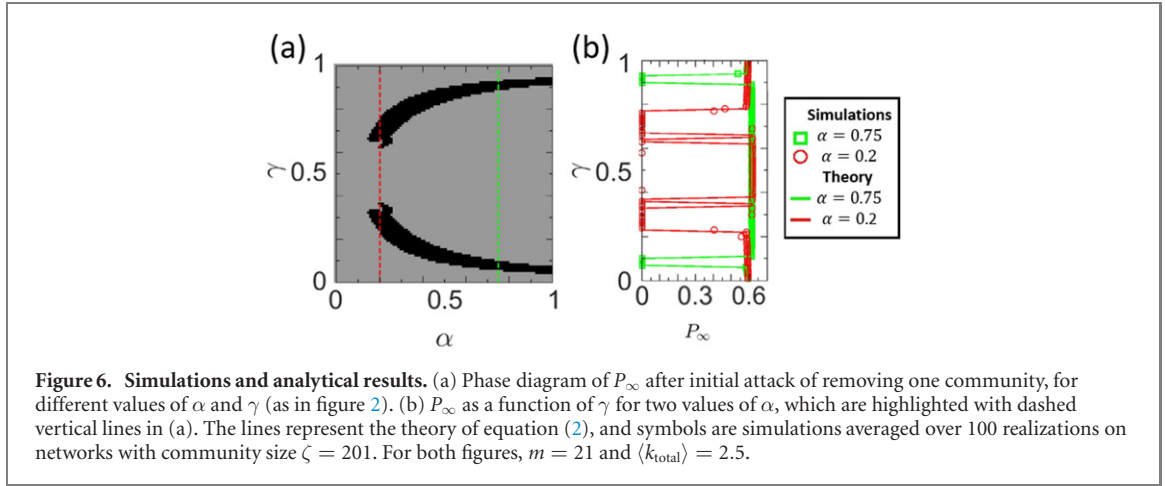
$$\Phi_{ij}^X(\vec{f}, \vec{p}) = 1 - p_j \left(1 - H_{ji}^X(f_{ji}) \prod_{\ell \in \Omega_j, \ell \neq i} G_{j\ell}^X(f_{j\ell}) \right) \quad (4)$$

and

$$\Psi_i^X(\vec{f}, \vec{p}) = p_i \left(1 - \prod_{j \in \Omega_i} G_{ij}^X(f_{ij}) \right). \quad (5)$$

Then for a multiplex model with two layers A and B , the vector equations of the cascading failures starting from $t = 0$ are:

$$\begin{aligned} \vec{f}_A(2t) &= \vec{\Phi}_A[\vec{f}_A(2t), \vec{p}(2t)], \\ \vec{p}(2t+1) &= \vec{\Psi}_A[\vec{f}_A(2t), \vec{p}(0)], \\ \vec{f}_B(2t+1) &= \vec{\Phi}_B[\vec{f}_B(2t+1), \vec{p}(2t+1)], \\ \vec{p}(2t+2) &= \vec{\Psi}_B[\vec{f}_B(2t+1), \vec{p}(0)]. \end{aligned} \quad (6)$$



Note that these equations can be generalized for more than two layers. For our anisotropic multiplex model, in which the degree distributions are Poisson distributions, such that $\langle k_{ij}^A \rangle = \langle k_{ij}^B \rangle = \langle k_{ij} \rangle$, the functions $\vec{\Phi}$ and $\vec{\Psi}$ are given by

$$\begin{aligned}\Phi_j(\vec{f}, \vec{p}) &\equiv 1 - p_j \left(1 - e^{-\sum_{i \in \Omega_j} \langle k_{ij} \rangle \cdot (f_i - 1)} \right) \\ \Psi_j(\vec{f}, \vec{p}) &\equiv p_j \left(1 - e^{-\sum_{i \in \Omega_j} \langle k_{ij} \rangle \cdot (f_i - 1)} \right),\end{aligned}\tag{7}$$

where $f_j \equiv f_{ij}$ and $\langle k_{ij} \rangle$ is defined in equation (3) in the main text.

Appendix B. The MGC equations for a multiplex lattice of communities

In a steady state at the end of the cascade, i.e. when $t \rightarrow \infty$, equation (6) give the MGC size of community i , $P_{\infty i}$:

$$\begin{aligned}1 - f_{ij}^A &= p_j \left[1 - H_{ji}^A(f_{ji}^A) \prod_{\ell \in \Omega_j, \ell \neq i} G_{j\ell}^A(f_{j\ell}^A) \right] \cdot \left[1 - \prod_{\ell \in \Omega_j} G_{j\ell}^B(f_{j\ell}^B) \right], \\ 1 - f_{ij}^B &= p_j \left[1 - H_{ji}^B(f_{ji}^B) \prod_{\ell \in \Omega_j, \ell \neq i} G_{j\ell}^B(f_{j\ell}^B) \right] \cdot \left[1 - \prod_{\ell \in \Omega_j} G_{j\ell}^A(f_{j\ell}^A) \right], \\ P_{\infty i} &= p_i \left[1 - \prod_{j \in \Omega_i} G_{ij}^A(f_{ij}^A) \right] \left[1 - \prod_{j \in \Omega_i} G_{ij}^B(f_{ij}^B) \right].\end{aligned}\tag{8}$$

In our multiplex model, we set the degree distributions to be the same in both layers. When assuming infinitely large ER communities model, all the distributions are Poisson and we obtain equation (2). It is noteworthy that other types of networks can be similarly solved by using the appropriate degree distributions. For example, for scale-free networks, the degree distributions of the intra-degree are $P_{ii}(k) = \frac{k^{-\alpha}}{\zeta(\alpha)}$, where $\zeta(\alpha)$ is the Riemann zeta function which is used as a normalizing constant.

Appendix C. Comparison between simulation results and theory

In figure 6, we compare between the theory equation (2) and numerical simulations for P_∞ after initial removal of one community and observe excellent agreement between them.

Appendix D. The analytical result for different system size

Figures 7 and 8 show that there is no effect of the system size m on our analytical results.

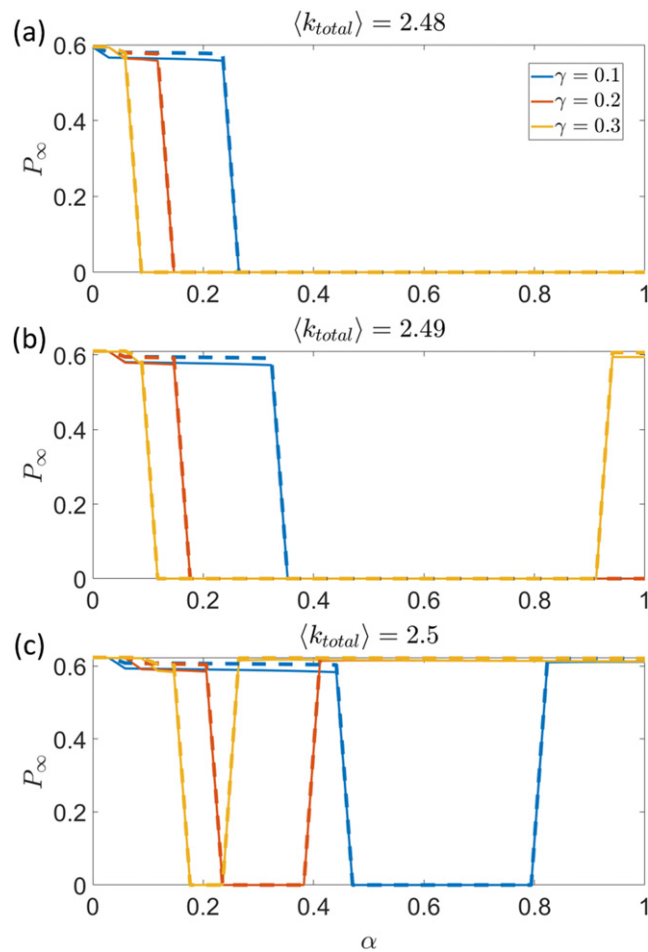


Figure 7. Robustness after removing one community for two values of m . (a)–(c) The functioning component, P_∞ , at steady state after initial removal of one community for different average total degree $\langle k_{\text{total}} \rangle$. The dashed lines show the P_∞ for the case of $m = 41$ and the continuous line is for the case of $m = 21$. The regions in which the system is stable ($P_\infty \approx 0.6$) and unstable ($P_\infty = 0$) do not depend on the size of the system.

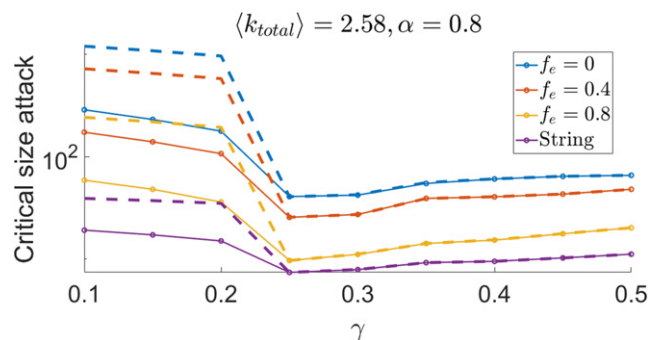


Figure 8. Comparing attacks of different shapes for two values of m . The critical size attack for attacks in an ellipse form and for removing a strip, on a log-linear graph, for a wide range of γ for $\langle k_{\text{total}} \rangle = 2.58$ and $\alpha = 0.8$ (as in figure 5(b)). The continuous lines represent $m = 21$ and the dashed lines represent $m = 41$. For $\gamma < 0.25$ the system size m affects the size of the critical attack, which is the whole system. In contrast, for $\gamma \geq 0.25$ the critical size does not depend on the system size m .

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