Novel Approach to Analysis of Nonlinear Recursions

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We propose a general method to map nonlinear recursions on a linear (but matrix) one. The solution of the recursion is represented as a product of matrices whose elements depend only on the form of the recursion and not on the initial conditions. First we consider the method for polynomial recursions of arbitrary order. Then we discuss the generalizations of the method for systems of polynomial recursions and arbitrary analytic recursions as well. The only restriction for these equations is to be solvable with respect to the highest order term (existence of a *normal* form).

I. INTRODUCTION

Recursions take a central place in various fields of science. Numerical solution of differential equations and different models of evolution of a system involve, in general, recursions.

By now, only linear recursions with constant coefficients could be solved [1–3]. However, the nonlinearity changes the situation drastically. Solution of a rather simple recursion, the *logistic map*,

$$y_{n+1} = \lambda y_n (1 - y_n)$$

is far from being so simple as one might guess. The analysis of its behavior, while based on a roundabout approach, has revealed many amazing properties.

In this paper we suggest a new approach to the solution of *nonlinear* recursions. It turns out, that the coefficients of the i-th iteration of the polynomial depend linearly on the coefficients of the $(i-1)$-th iteration. Using this fact we succeed to write down the general solution of the polynomial recursion and to generalize it for any recursion

$$y_{n+1} = f(y_n) \qquad \text{with} \quad y_0 \equiv y,$$

where $f(x)$ is an analytic function. In the paper we study the conditions that the analytic function must satisfy to make this generalization possible.

The gist of the method is the construction of a special *transfer matrix,* **T**. It allows to represent the solution in the form

$$y_n = \langle \mathbf{e} | \mathbf{T}^n | \mathbf{y} \rangle, \tag{1}$$

where $\langle \mathbf{e}|$ is the first unit vector, $|\mathbf{y}\rangle$ is a vector of initial values and $\mathbf{T}^n$ is the n-th power of the matrix **T**.

II. FIRST-ORDER POLYNOMIAL RECURSION

Here we consider a first-order recursion equation in its *normal* form

$$y_{n+1} = P(y_n)\,, \tag{2}$$

where $P(x)$ is a polynomial of degree m:

$$P(x) = \sum_{k=0}^{m} a_k x^k\,, \qquad a_m \neq 0. \tag{3}$$

Let $y_0 \equiv y$ be an initial value for the recursion (2). We denote by $|\mathbf{y}\rangle$ the column vector of powers of y $|\mathbf{y}\rangle = \{y^j\}_{j=0}^{\infty}$ and the vector $\langle \mathbf{e}|$ is a row vector $\langle \mathbf{e}| = [\delta_{j1}]_{j=0}^{\infty}$. It should be emphasized, that j runs from 0, since in the general case $a_0 \neq 0$. In this notation $\langle \mathbf{e}|\mathbf{y}\rangle$ is a scalar product that yields

$$\langle \mathbf{e}|\mathbf{y}\rangle = y\,. \tag{4}$$

Theorem 1. *For any recursion of type Eq. (2) there exists a matrix* $\mathbf{T} = \{T_{jk}\}_{j,k=0}^{\infty}$ *such that*

$$y_n = \langle \mathbf{e}|\mathbf{T}^n|\mathbf{y}\rangle. \tag{5}$$

The matrix power $\mathbf{T}^n$ *exists for all* n *and all the operations in the right-hand side of Eq. (5) are associative.*

Proof. For $n = 0$ the statement of the theorem is valid (see Eq. (4)). We introduce the column vector $|\mathbf{y}_1\rangle \stackrel{\text{def}}{=} \left\{y_1^j\right\}_{j=0}^{\infty}$, where $y_1 = P(y)$. Let $\mathbf{T}$ be a matrix such that

$$|\mathbf{y}_1\rangle = \mathbf{T}|\mathbf{y}\rangle\,. \tag{6}$$

The existence of this matrix will be proven later on. If such a matrix exists, then, analogically to Eq. (4), we have $y_1 = \langle \mathbf{e}|\mathbf{y}_1\rangle = \langle \mathbf{e}|\mathbf{T}|\mathbf{y}\rangle$. Therefore, the statement of the theorem is true for $n = 1$ as well.

Assume, that Eq. (5) is valid for $n = l$ for any initial value y. Then y_{l+1} can be represented as $y_{l+1} = \langle \mathbf{e}|\mathbf{T}^l|\mathbf{y}_1\rangle$, where $y_1 = P(y)$ is considered as a new initial value of the recursion. Then, using Eq. (6) one gets

$$y_{l+1} = \langle \mathbf{e}|\mathbf{T}^l|\mathbf{y}_1\rangle = \langle \mathbf{e}|\mathbf{T}\mathbf{T}^l|\mathbf{y}\rangle = \langle \mathbf{e}|\mathbf{T}^{l+1}|\mathbf{y}\rangle\,.$$

Now we prove the existence of the matrix $\mathbf{T}$. One has $\langle \mathbf{y}_1| \stackrel{\text{def}}{=} [P^j(y)]_{j=0}^{\infty}$. In turn $P^j(y)$ is the jm-th degree polynomial

$$P^j(y) = \left(\sum_{i=0}^{m} a_i y^i\right)^j = \sum_{k=0}^{jm} T_{jk} y^k, \tag{7}$$

and we infer that $\mathbf{T} = \{T_{jk}\}_{j,k=0}^{\infty}$ obeys Eq. (6).

Note, that for j and k satisfying $k \geq jm$ we have $T_{jk} \equiv 0$, therefore, each row is finite (*i.e.*, there is only a finite number of nonzero matrix elements in each row). This proves the existence of powers of $\mathbf{T}$ and associativity in Eq. (5). Thus, the proof is complete.

III. SPECIAL CASES

In some special cases the matrix $\mathbf{T}$ has the rather simple form.

(a) The binomial: $P(x) = a_p x^p + a_q x^q$. As one can see, in the general case elements of the matrix $\mathbf{T}$ have a form of rather complicated sums. However, they are degenerated to a fairly simple expression, when the polynomial (3) has only two terms. In this case one has

$$P^j(y) = (a_p y^p + a_q y^q)^j = \sum_{i=0}^{j} \binom{j}{i} a_p^{j-i} a_q^i y^{p(j-i)+qi}.$$

Denoting $k = p(j-i) + qi$, $i = l(k) = (q-p)^{-1}(k - pj)$, we have

$$P^j(y) = \sum_{k=jp}^{jq} y^k \binom{j}{l(k)} a_p^{j-l(k)} a_q^{l(k)}.$$

Thus, the matrix elements T_{jk} are

$$T_{jk} = \binom{j}{l(k)} a_p^{j-l(k)} a_q^{l(k)}. \tag{8}$$

Example 1. To demonstrate this let us consider the following recursion:

$$y_{n+1} = \lambda y_n (1 - y_n) \quad \text{with} \quad y_0 \equiv y\,. \tag{9}$$

Plugging $p = 1$, $q = 2$, $a_p = -a_q = \lambda$ into Eq. (8) one immediately obtains

$$T_{jk} = (-1)^{k-j} \binom{j}{k-j} \lambda^j\,. \tag{10}$$

Thus, we recover the result of Rabinovich *et al* [4] for the recursion equation known as the *logistic mapping*.

(b) The trinomial $P(x) = a_0 + a_p x^p + a_q x^q$ and $a_0 \neq 0$. Then the matrix $\mathbf{T}$ has a special form.

Lemma 1. *Let* $P(x) = a_0 + a_p x^p + a_q x^q$. *Then, the following decomposition exists*

$$\mathbf{T} = \mathbf{A}\mathbf{T}_0\,,$$

where $\mathbf{T}_0$ *is a matrix corresponding to the polynomial* $P_0(x) = a_p x^p + a_q x^q$ *and* $\mathbf{A}$ *is a triangular matrix.*

Proof. Consider $P_0(x) = a_p x^p + a_q x^q$ and the corresponding matrix $\mathbf{T}_0$. It yields

$$\mathbf{T}_0|\mathbf{y}\rangle = |\mathbf{y}_1'\rangle \stackrel{\text{def}}{=} \left\{P_0^j(y)\right\}_{j=0}^{\infty} .$$

For the matrix $\mathbf{T}$ one gets $\mathbf{T}|\mathbf{y}\rangle = |\mathbf{y}_1\rangle \stackrel{\text{def}}{=} \{P(y)^j\}_{j=0}^{\infty}$,

$$P^j(y) = \sum_{i=0}^{j} \binom{j}{i} a_0^{j-i} \left(a_p y^p + a_q y^q\right)^i .$$

Denoting in the last line $A_{ji} \equiv \binom{j}{i} a_0^{j-i}$ one obtains $|\mathbf{y}_1\rangle = \mathbf{A}|\mathbf{y}_1'\rangle = \mathbf{A}\mathbf{T}_0|\mathbf{y}\rangle = \mathbf{T}|\mathbf{y}\rangle$, and $\mathbf{T} = \mathbf{A}\mathbf{T}_0$. The lemma is proven.

IV. POLYNOMIAL RECURSIONS WITH NON-CONSTANT COEFFICIENTS

As shown in [4] a generalization of Eq. (9),

$$y_{n+1} = \lambda_n y_n (1 - y_n) \quad \text{with} \quad y_0 \equiv y\,, \tag{11}$$

can be solved using a similar approach. The solution is

$$y_n = \langle \mathbf{e}|\mathbf{T}_n \cdots \mathbf{T}_2 \mathbf{T}_1|\mathbf{y}\rangle\,, \tag{12}$$

where the matrix elements of $\mathbf{T}_i$ are now i-dependent:

$$(T_i)_{jk} = (-1)^{k-j} \binom{j}{k-j} (\lambda_i)^j. \tag{13}$$

The same argument is valid for the general case.

Theorem 2. *Let the polynomial in the Eq. (2) depends on n. Then the solution Eq. (5) takes the form of Eq. (12) with the obvious changes ($a_0, \ldots, a_m$ become i-dependent functions) in corresponding matrix elements.*

To illustrate this Theorem we consider

Example 2. We are going to apply the previous material to the *Riccati equation.* Usually, people use this name for the equation

$$y_{n+1} y_n + a_n' y_{n+1} + b_n' y_n + c_n' = 0.$$

However, by appropriate variable change [1,2] this equation can be reduced to a linear one and then treated by conventional techniques. Here we shall be dealing with the following recursion:

$$y_{n+1} = a_n + b_n y_n + c_n y_n^2 \quad \text{with} \quad y_0 \equiv y\,.$$

This is another possible (asymmetric) discrete analog of the Riccati differential equation [5–7]. It is well-known that the latter cannot be solved in quadratures.

The general results of the two previous sections can be employed to write down the solution of this recursion. Namely, the solution reads

$$y_n = \langle \mathbf{e} | \mathbf{T}_n \cdots \mathbf{T}_2 \mathbf{T}_1 | \mathbf{y} \rangle\,,$$

where the matrix $\mathbf{T}_i$ is a product of two matrices

$$\mathbf{T}_i = \mathbf{A}_i \mathbf{S}_i\,.$$

The matrix elements are given by

$$(\mathbf{A}_i)_{jk} = \binom{j}{k} a_i^{j-k} \quad \text{and} \quad (\mathbf{S}_i)_{jk} = \binom{j}{k-j} b_i^{2j-k} c_i^{k-j}\,.$$

V. SYSTEM OF FIRST-ORDER NONLINEAR EQUATIONS

Actually, very little is known about systems of nonlinear equations [8]. We now extend our method of Sect. II to handle systems of nonlinear equations. Let us demonstrate it on the following example:

$$\begin{cases} u_{n+1} &= \lambda u_n (1 - v_n) \quad \text{with} \quad u_0 \equiv u\,, \\ v_{n+1} &= \mu v_n (1 - u_n) \quad \text{with} \quad v_0 \equiv v\,. \end{cases} \tag{14}$$

Proceeding here as in Sect. II, we are checking the transformation of a product $u^j v^k$:

$$\begin{aligned} [\lambda u(1-v)]^j [\mu v(1-u)]^k &= \sum_{r,s} \lambda^j u^j (-1)^r \binom{j}{r} v^r \mu^k v^k (-1)^s \binom{k}{s} u^s \\ &= \sum_{p,q} u^p v^q (-1)^{(p-j)+(q-k)} \binom{j}{q-k} \binom{k}{p-j} \lambda^j \mu^k\,. \end{aligned} \tag{15}$$

One can proceed with the aid of multidimensional matrices [11], but here we prefer to use more traditional two-dimensional matrices.

Introduce now a vector $|\mathbf{uv}\rangle$, which consists of powers $u^j v^k$ of variables u and v, arranged in some order. Namely, introduce a bijection $(\cdot,\cdot)\colon \mathbf{N}^2 \to \mathbf{N}$, where $\mathbf{N}$ is the set of the natural numbers and zero. Then, the monomial $u^j v^k$ is the (j,k)-th component of the vector $|\mathbf{uv}\rangle$. For example, one can use

$$(j,k) = k + \frac{1}{2}(j+k)(j+k+1).$$

Introducing a matrix $\mathbf{T}$ with the elements

$$T_{(j,k)(p,q)} = (-1)^{(p-j)+(q-k)} \binom{j}{q-k} \binom{k}{p-j} \lambda^j \mu^k$$

we basically return to the familiar transfer-matrix construction. Indeed, we have

$$u_n = \langle \mathbf{e}_{(1,0)} | \mathbf{T}^n | \mathbf{uv} \rangle,$$

$$v_n = \langle \mathbf{e}_{(0,1)} | \mathbf{T}^n | \mathbf{uv} \rangle,$$

where $\langle \mathbf{e}_{(j,k)}| = \{\delta_{(j,k),i}\}_{i=0}^{\infty}$.

Theorem 3. *Consider a system of m first-order nonlinear equations:*

$$x_{n+1}^{(i)} = P_i(x_n^{(1)}, \ldots, x_n^{(m)}), \qquad i = 1, \ldots, m. \tag{16}$$

Then the matrix $\mathbf{T}$ exists such that

$$x_n^{(i)} = \langle \mathbf{e}^{(i)} | \mathbf{T}^n | \mathbf{x^{(1)}} \cdots \mathbf{x^{(m)}} \rangle,$$

where the vectors $\langle \mathbf{e}^{(i)}|$ and $|\mathbf{x^{(1)}} \cdots \mathbf{x^{(m)}}\rangle$ are defined as above, with the aid of some bijection $(\cdot, \ldots, \cdot)\colon \mathbf{N}^m \to \mathbf{N}$.

Below, we present the scheme of the proof of a transfer-matrix representation for arbitrary systems of first-order nonlinear equations.

As before we are checking the product

$$P_1^{j_1}\cdots P_m^{j_m}\equiv P_{(j_1,\ldots,j_m)}.$$

The polynomial $P_{(j_1,\ldots,j_m)}$ depends on m variables $x^{(1)},\ldots,x^{(m)}$ and therefore can be represented as

$$P_{(j_1,\ldots,j_m)}(x^{(1)},\ldots,x^{(m)})=\langle \mathbf{t}_{(j_1,\ldots,j_m)}|\mathbf{x^{(1)}}\cdots\mathbf{x^{(m)}}\rangle,$$

where $\langle \mathbf{t}_{(j_1,\ldots,j_m)}|$ is the constant vector of coefficients of the polynomial $P_{(j_1,\ldots,j_m)}$. This vector $\langle \mathbf{t}_{(j_1,\ldots,j_m)}|$ is the $(j_1,\ldots,j_m)$-th row of the transfer matrix $\mathbf{T}$.

The above procedure can be easily generalized on polynomials with variable coefficients, $P_i(n,x^{(1)},\ldots,x^{(m)})$, in complete analog to the Theorem 2.

VI. RECURSION SOLUTION FOR ARBITRARY ANALYTIC FUNCTIONS IN THE RHS

Let the series

$$\sum_{k=0}^{\infty} a_k x^k \equiv f(x) \tag{17}$$

converge absolutely in $[-r,r]$. Then the series

$$\sum_{k=0}^{\infty} \tilde{a}_k x^k \equiv \tilde{f}(x), \tag{18}$$

where $\tilde{a}_k=|a_k|$, also converge in $[-r,r]$. We construct the following absolutely convergent in $[-r,r]$ [9,10] series

$$\left[\sum_{k=0}^{\infty} a_k x^k\right]^j = a_0^j+\binom{j}{1}a_1a_0^{j-1}x+\left[\binom{j}{1}a_2a_0^{j-1}+\binom{j}{2}a_1^2a_0^{j-2}\right]x^2+\ldots=\sum_{k=0}^{\infty}T_{jk}x^k, \tag{19}$$

$$\left[\sum_{k=0}^{\infty} \tilde{a}_k x^k\right]^j = \tilde{a}_0^j+\binom{j}{1}\tilde{a}_1\tilde{a}_0^{j-1}x+\left[\binom{j}{1}\tilde{a}_2\tilde{a}_0^{j-1}+\binom{j}{2}\tilde{a}_1^2\tilde{a}_0^{j-2}\right]x^2+\ldots=\sum_{k=0}^{\infty}\tilde{T}_{jk}x^k. \tag{20}$$

Definition 1. *The matrix* $\mathbf{T}=\{T_{jk}\}_{j,k=0}^{\infty}$ *is said to be the transfer matrix of the analytic function* $f(x)$ *if* $\sum_{k=0}^{\infty}T_{jk}x^k=f^j(x)$.

Using Eq. (19) one can estimate the coefficients T_{jk} for a fixed k:

$$|T_{jk}| \le \binom{j+k}{k} a_0^{j-k} \left[\max_{i\le k} a_i\right]^k \le C(a_0+\epsilon)^j \tag{21}$$

for arbitrary $\epsilon > 0$, a constant $C = C(\epsilon)$, and $j > k$.

Theorem 4. *Let the series*

$$\sum_{k=0}^{\infty} b_k x^k \equiv g(x) \tag{22}$$

converge absolutely in $[-R, R]$*. The matrix* $\mathbf{S}$ *be the transfer matrix of the function* $g(x)$*, and* $\tilde{r} \le r$ *be such that the function* $\tilde{f}(x)$*, defined by* (18)*, satisfies* $\tilde{f}(|x|)\colon [-\tilde{r}, \tilde{r}] \to [0, R]$*. Then the matrix product* $\mathbf{ST}$ *exists and is equal to the transfer matrix of the composition of the functions* $f(x)$ *and* $g(x)$*,* $g \circ f(x)$*, i.e., the* j*-th row of the matrix* $\mathbf{ST}$ *is the vector of the coefficients of Maclaurin's decomposition of the function* $[g \circ f]^j(x)$*.*

Proof. First we prove the existence of the matrix $\mathbf{ST}$. One can estimate the (j, k)-th element of the matrix $\mathbf{ST}$ using Eq. (21):

$$|(ST)_{jk}| \le \sum_{i=0}^{\infty} |S_{ji}||T_{ik}| \le C_1 + \sum_{i=k+1}^{\infty} |S_{ji}||T_{ik}| \le C_1 + C \sum_{i=k+1}^{\infty} |S_{ji}|(a_0+\epsilon)^i.$$

Since $\min \tilde{f}(x) = \tilde{f}(0) = a_0 < R$ we can choose $\epsilon > 0$ sufficiently small, that $a_0 + \epsilon < R$, therefore the series in the last expression converges. Thus, the matrix $\mathbf{ST}$ exists.

Now we should study the series

$$\sum_{k=0}^{\infty} (ST)_{jk} x^k = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} S_{ji} T_{ik} x^k. \tag{23}$$

Since the series converge absolutely:

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} |S_{ji} T_{ik} x^k| \le \sum_{i=0}^{\infty} |S_{ji}| \sum_{k=0}^{\infty} \tilde{T}_{ik} |x^k| \le \sum_{i=0}^{\infty} |S_{ji}| \left[\tilde{f}(|x|)\right]^i < \infty. \tag{24}$$

we can change the order of summation [10]:

$$\sum_{k=0}^{\infty} (ST)_{jk} x^k = \sum_{i=0}^{\infty} S_{ji} \sum_{k=0}^{\infty} T_{ik} x^k = \sum_{i=0}^{\infty} S_{ji} \left[f(x)\right]^i = \left[g(f(x))\right]^j = \left[g \circ f\right]^j (x).$$

The theorem is proved.

Restricting ourselves to the first row of the matrix $\mathbf{S}$ only we get the following

Corollary 1. *Let the functions $f(x)$ and $g(x)$ satisfy the conditions of* Theorem 4; *the bra-vector $\langle \mathbf{b}|$ is the vector of coefficients of the series* (22). *Then the vector $\langle \mathbf{c}| = \langle \mathbf{b}|\mathbf{T}$ consists of the coefficients of the series*

$$\sum_{k=0}^{\infty} c_k x^k = g \circ f(x). \tag{25}$$

It is easy to see, that calculating the scalar product of the vector $\langle \mathbf{c}|$ and the vector $|\mathbf{x}\rangle = \{x^i\}_{i=0}^{\infty}$ we obtain

$$\langle \mathbf{b}|\mathbf{T}|\mathbf{x}\rangle = g \circ f(x). \tag{26}$$

In this way we can write down the solution of the recursion

$$y_{n+1} = f(y_n) \qquad \text{with } y_0 \equiv y. \tag{27}$$

Theorem 5. *Let the series* (17) *converge absolutely on $[-r, r]$, the matrix $\mathbf{T}$ be the transfer matrix of the function $f(x)$ and $\tilde{r} < r$ be such that* (see Eq. (18)) $\tilde{f}(|x|)\colon [-\tilde{r}, \tilde{r}] \to [0, \tilde{r}]$. *Then for any initial conditions $y \in [-\tilde{r}, \tilde{r}]$*

$$y_n = \langle \mathbf{e}|\mathbf{T}^n|\mathbf{y}\rangle, \tag{28}$$

where the bra-vector $\langle \mathbf{e}| = [\delta_{i,1}]_{i=0}^{\infty}$ and the ket-vector $|\mathbf{y}\rangle = \{y^i\}_{i=0}^{\infty}$.

In order to solve a generalization of the recursion (27), $y_{n+1} = f(n, y_n)$, we should consider n-dependent matrices $\mathbf{T}_n$ instead of the matrix $\mathbf{T}$.

One can use the same technique to solve the more general case of multivariable recursions,

$$\mathbf{y}_n = f(\mathbf{y}_{n-1}), \qquad f\colon \mathbf{R}^n \to \mathbf{R}^n, \quad f(\mathbf{x}) = \{f_i(\mathbf{x})\}_{i=1}^{n}, \quad \mathbf{x} \in \mathbf{R}^n$$

Here we should consider series decomposition of the functions $\prod_{i=1}^{n} [f_i(\mathbf{x})]^{a_i}$, instead of the series (19), and then proceed by the analogy to multivariable polynomial recursions. However, since the notation becomes very cumbersome, we shall not write down this formalism.

The theorems of this section can be easily generalized on complex mappings. Namely, the statement of the Theorem 4 for the complex variable functions reads:

Theorem 6. *Let the series (17), (18) and (22) be the series of complex variable z with the complex coefficients. Series (17) and (18) converge in the circle $|z| \leq r$ and series (22) converge in $|z| \leq R$. The matrices $\mathbf{S}$ and $\mathbf{T}$ be the transfer matrices of the functions $g(z)$ and $f(z)$ correspondingly and $\tilde{r} \leq r$, $\tilde{r} \in \mathbf{R}$ be such that $\tilde{f}(\tilde{r}) \leq R$. Then the matrix product $\mathbf{ST}$ exists and equals to the transfer matrix of the composition of the functions $f(z)$ and $g(z)$, $g \circ f(z)$, i.e., the j-th row of the matrix $\mathbf{ST}$ is the vector of the coefficients of Maclaurin's decomposition of the function $[g \circ f]^j(z)$ convergent in $|z| \leq \tilde{r}$.*

VII. EXAMPLES

In order to show that the condition $\tilde{f}(|x|)\colon [-\tilde{r}, \tilde{r}] \to [0, R]$ is essential (see Theorem 4) we present the following

Example 3. Let

$$g(x) = x^4 - \frac{1}{4}x^6 + \frac{1}{9}x^8 - \frac{1}{16}x^{10} + \dots,$$

i.e. $a_{2i} = \frac{(-1)^i}{(i-1)^2}$, $a_{2i+1} = 0$, $i \geq 2$ and radius of convergence is $R = 1$; let $f(x)$ be defined by $f(x) = 2x(1-x^2)$, $f\colon [-1,1] \to [-1,1]$. The transfer matrix $\mathbf{T}$ of the function $f(x)$ according to Eq. (8) is $T_{jk} = (-1)^{(k-j)/2}\binom{j}{(k-j)/2}2^j$ for even $j-k$. The sign in the sequences $\{S_{1,2i}\}_{i=1}^{\infty}$ and $\{T_{2i,2k}\}_{i=1}^{\infty}$ alternates. Therefore the sign in the sequence $\{S_{1,2i}T_{2i,2k}\}_{i=1}^{\infty}$ is constant and for the $2k$-th component of the first row of the matrix $\mathbf{ST}$ one has

$$\begin{aligned} |c_{1,2k}| &= \left|\sum_{i=0}^{\infty} S_{1,2i}T_{2i,2k}\right| = \sum_{i=0}^{\infty} |S_{1,2i}T_{2i,2k}| \geq |S_{1,2(k-2)}T_{2(k-2),2k}| \\ &= 4^{k-2}\frac{1}{(k-3)^2}\frac{(2k-4)(2k-5)}{2} \geq 4^{k-2} \end{aligned}$$

for any $k > 3$. Thus, the components of the the first row cannot be the coefficients of Maclaurin's decomposition of the function $g \circ f$ because of its divergence for $x > 1/4$.

The following example presents a family of function whose transfer matrices form is invariant under the multiplication.

Example 4. Let the function $f(x) = ax/(b+x)$. Then the components of the transfer matrix $\mathbf{T}$ are

$$T_{jk} = \binom{k-1}{j-1} a^j b^{-k}, \quad \text{for } j,k > 0 \quad \text{and} \quad T_{jk} = \delta_{0j}\delta_{0k}, \quad \text{for } j=0 \text{ or } k=0, \tag{29}$$

where

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!} & \text{integer } m,n \quad 0 \le m \le n, \\ 0 & \text{any other } m,n; \end{cases}$$

and δ_{nm} is the Kronecker symbol. If $g(x) = cx/(d+x)$ and $\mathbf{S}$ is the transfer matrix of $g(x)$, then the components of the matrix $\mathbf{ST}$ are

$$\begin{aligned}(ST)_{jk} &= \sum_{i=0}^{\infty} S_{ji}T_{ik} = \sum_{i=0}^{\infty} \binom{i-1}{j-1} c^j d^{-i} \binom{k-1}{i-1} a^i b^{-k} = \binom{k-1}{j-1} c^j b^{-k} \sum_{l=0}^{k-j} (a/d)^{l+j} \binom{k-j}{l} \\ &= \binom{k-1}{j-1} \left(\frac{ac}{a+b}\right)^j \left(\frac{bd}{a+b}\right)^{-k}.\end{aligned}$$

One can see, that the matrix elements $(ST)_{jk}$ have the same form as in Eq. (29).

VIII. SUMMARY

In this paper we have presented a new method to obtain the solution of arbitrary polynomial recursions. The method has been generalized to the systems of multivariable recursions and analytical recursions. We found a class of analytical function recursions to which the method can be applied. Particularly, this class contains functions which are analytical over all space.

Generally, the solution is obtained in the form of a matrix power, applied to the vectors of initial values. We have presented a way to construct such a matrix.

Famous and important examples, such as the *logistic map* and the *Riccati recursion*, have been considered and the corresponding matrices have been written down explicitly.

The following generalizations are also can be done:

a) Multivariable analytic recursions. This can be done, for example, as in the Section V.

b) System of higher-order nonlinear equations. The scheme is quite obvious: introduction of new variables to bring each equation to the first-order structure and, then, construction of a transfer matrix, as in the previous case.

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