J. Phys. A: Math. Gen. 18 (1985) L1103-L1107. Printed in Great Britain

LETTER TO THE EDITOR

Family of growth fractals with continuously tunable chemical dimension

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Received 27 August 1985

Abstract. We introduce a new class of statistical growth fractals which is of interest because the chemical dimension d_l is continuously tunable. We also study other exponents characterising these fractals.

There has been considerable recent interest in uncovering the fashion in which the familiar laws of physics are modified for fractal objects, in part because of the large number of important realisations of fractals in nature [1-8]. It has only recently become appreciated that the physics of fractals is determined by more than just 'the' fractal dimension d_f which describes how the cluster mass N scales with the cluster radius R,

$$N \sim R^{d_f}. \tag{1a}$$

Several additional fractal dimensions have recently been found to be of use. One of these is the chemical dimension d_l that describes how the cluster mass scales within a chemical distance l (the chemical distance is the length of the shortest path on the fractal connecting two sites) [9-14],

$$N \sim l^{d_l}. \tag{1b}$$

It is of course very important to seek relations among the various new fractal dimensions—indeed, it was the search for relations among critical exponents that led to the discovery of scaling laws 20 years ago and eventually to the development of the renormalisation group.

Here we study the relation between the two exponents d_f and d_l by introducing a new family of cluster growth models in which d_l can be varied in a controlled fashion over a wide range of values.

The clusters are grown by the following procedure. First we place a seed particle at the origin of a d-dimensional lattice of coordination number z. At step $l=1$, we randomly choose a certain number $B(1)$ of the z neighbours and occupy these sites. These sites constitute the first shell, and clearly have chemical distance $l=1$ from the origin. The remaining $z-B(1)$ sites will be regarded as 'blocked' *for the duration of the growth process.* At step $l=2$, we consider z_2 available (unblocked and unoccupied) neighbours of the first shell. We randomly occupy $B(2)$ of these, and block the

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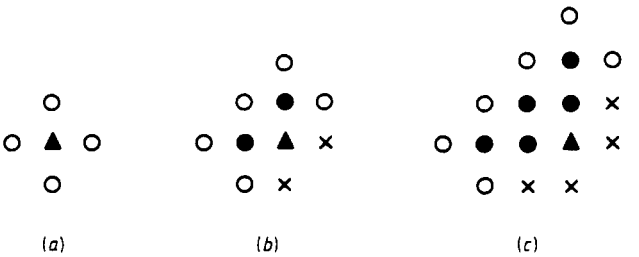

Figure 1. Schematic illustration of how a cluster grows for $B(l) = cl^{\alpha}$ with $c = 2$ and $\alpha = 0.5$, where $B(l)$ is the number of growing sites at shell l, and α is the tunable parameter. In figure 1(a), at step $l = 0$, the seed particle (▲) is at the origin of a square lattice with four perimeter sites (○). In figure 1(b), at step $l = 1$, we choose $B(1)$ ($=2\times1^{0.5}=2$) sites (●) randomly out of four perimeter sites of figure 1(a) and occupy them. The remaining sites (×) are blocked forever. Five new perimeter sites are created. In figure 1(c), at step $l = 2$, $B(2)$ ($=2\times2^{0.5}=3$) sites are added to randomly chosen perimeter sites (○) of figure 1(b) and the rest are blocked (×). Remaining shells grow similarly.

remainder. This process is continued until a cluster with a total of $l_{\max}$ chemical shells has been created, with

$$B(l) \sim l^{\alpha} \tag{2a}$$

occupied sites in chemical shell l; here α is the tunable parameter†. The total cluster mass after l shells have been added is given by

$$N(l) = \sum_{l'=1} B(l') \sim l^{d_l}, \tag{2b}$$

$$d_l = \alpha + 1. \tag{2c}$$

A schematic illustration of our cluster model is shown in figure 1, and a typical cluster is shown in figure 2.

We begin by considering how the Pythagorean distance R scales with the chemical distance l. From equations (1a, b) $R^{d_f} \sim l^{d_l}$, so

$$R \sim l^{\tilde{\nu}}, \tag{3a}$$

where

$$d_{\min} = \tilde{\nu}^{-1} = d_f / d_l. \tag{3b}$$

Figure 2. A typical cluster grown on a square lattice with $d_l = 1.2$.

† The case $\alpha = 0$ was considered recently [15].

Table 1. Data for $\tilde{\nu} = d_{\min}^{-1}$, d_w^l, d_w and $\frac{1}{2}d_s$ in two dimensions. $\tilde{\nu}$ is increasing and d_w is decreasing as d_l increases, but d_w^l has a maximum. The fracton dimension d_s is also monotonically increasing.

d_l	$\tilde{\nu} = d_{\min}^{-1}$	d_w^l	d_w	$\frac{1}{2}d_s$
1.0	0.67 ± 0.02	2.0	3.0 ± 0.1	0.5 ± 0.03
1.2	0.73 ± 0.03	2.2 ± 0.1	3.0 ± 0.1	0.55 ± 0.03
1.4	0.80 ± 0.03	2.37 ± 0.1	2.9 ± 0.1	0.60 ± 0.03
1.64	0.88 ± 0.03	2.30 ± 0.1	2.6 ± 0.1	0.7 ± 0.03
1.8	0.93 ± 0.03	2.29 ± 0.1	2.4 ± 0.1	0.77 ± 0.03

Table 2. Data for $\tilde{\nu}$ in $d = 3, 4$ and 5.

$d = 3$		$d = 4$		$d = 5$	
d_l	$\tilde{\nu} = d_{\min}^{-1}$	d_l	$\tilde{\nu} = d_{\min}^{-1}$	d_l	$\tilde{\nu} = d_{\min}^{-1}$
1.0	0.50 ± 0.01	1.0	0.5 ± 0.01	1.0	0.5 ± 0.01
1.5	0.67 ± 0.03	1.3	0.5 ± 0.03	1.5	0.5 ± 0.02
2.0	0.78 ± 0.03	1.5	0.56 ± 0.03		
2.5	0.90 ± 0.03	2.0	0.67 ± 0.03		
		2.5	0.73 ± 0.03		
		3.0	0.83 ± 0.03		

In order to study this new fractal family, we first calculate d_f and $d_{\min} = \tilde{\nu}^{-1}$ for a range of choices of d_l. The results are given in tables 1 and 2 for Euclidean dimensions $d = 2$-5. The case $\alpha = 0$ ($d_l = 1$) corresponds to chemically linear branched polymers [15]. The numerical values given in tables 1 and 2 were found for $l \leqslant 200$†, and there is reason to believe that they are not the true asymptotic exponents. For example, we know that this problem is characterised by a 'slow crossover' to the asymptotic regime since the case $d_l = 1$ corresponds [16] to the kinetic growth walk (KGW)‡. Indeed, for the case $d_l = 1$ and $d = 2$, our analysis suggests that $\tilde{\nu} = 1/d_{\min} = 0.67$ while we know [17] that for sufficiently large l, the KGW universality class crosses over to that of the self-avoiding random walk, for which $d_{\min} = d_f = 4/3$ *exactly* so that $\tilde{\nu} = 1/d_{\min} = 3/4$. This slow crossover has led to errors in interpreting previous numerical results [16, 18§], so there is little reason to expect that the same troubles are not present here, especially for $d > 2$ (e.g., the crossover for $d = 3$ is two orders of magnitude higher than for $d = 2$ since self trapping is rare).

For $d = 2$ percolation, $d_l \simeq 1.64$ [9-12]. Hence we studied in some detail the present model for the special case $\alpha = 0.64$. Of course, a fractal is characterised by as many as ten different fractal dimensions [19], so there is no *a priori* reason to expect that percolation and the $d_l = 1.64$ case of the present model are identical. Nonetheless, we found that $\tilde{\nu} = 1/d_{\min} = 0.88$ is roughly the same value as percolation [9-12], and we also found that the ratio of perimeter to mass is close to what one would expect from

† It is difficult to generate clusters for large l due to self trapping.

‡ The kinetic growth walk (KGW) is the unbranched limit of the present model, $B(l) = 1$ for all l [16].

§ See also the recent discussion of the 'slow crossover' problem [18a].

percolation, $(1-p_c)/p_c \simeq 407/593$. Of course, the two models cannot be completely identical since chemical shell l has *exactly* l^{d_l-1} cluster sites in the present model, while in percolation chemical shell l has only l^{d_l-1} sites on *average*. This difference is dramatic if one considers the application of d_l to the quantitative analysis of a forest fire [23] set at time $t=0$ on a percolation cluster. If each tree ignites only its neighbours on successive time steps, and if each tree burns for one unit of time, then in the present model there are *precisely* l^{d_l-1} burning trees at time t, while in percolation there are an *average* of l^{d_l-1} burning trees. The dispersion about this average is a topic of present investigation.

Next we consider the subtle question of loops. We know that for the special case $\alpha=0$ ($d_l=1$), loops are irrelevant [15]. For small α ($\alpha<0.2$), loops do not *visually* appear on all length scales, while for large α they do. Hence it is tempting to suggest that this model has the intriguing feature of a 'loop threshold' $\alpha=\alpha_c$, below which loops are irrelevant. The critical properties of the loops near the threshold $\alpha=\alpha_c$ is a topic for further study.

In order to study dynamic exponents, we performed *exact* enumeration [14, 20] of diffusion on these clusters for $d=2$. We calculated the diffusion exponent d_w

$$t \sim R^{d_w}, \tag{4a}$$

the 'chemical-space' diffusion exponent d_w^l

$$t \sim l^{d_w^l}, \tag{4b}$$

and the fracton (spectral) dimension d_s

$$P_0 \sim t^{-d_s/2}. \tag{4c}$$

Here t is the time and P_0 is the probability of returning to the origin. The results for the exponents are presented in table 1. We note that these values are consistent with the relations [10, 13]

$$\tilde{\nu} = d_l/d_f = d_w^l/d_w, \tag{5a}$$

and

$$\tfrac{1}{2}d_s = d_f/d_w = d_l/d_w^l. \tag{5b}$$

The data for small values of d_l are also consistent with [13]

$$d_w^l = d_l+1, \qquad d_w = d_f(1+1/d_l), \qquad d_s = 2d_l/(d_l+1), \tag{5c}$$

which hold when loops are not relevant, suggesting that loops may not occur on all length scales for small d_l†.

It should be noted that the upper critical dimension d_c for the present growth model is bounded from above by $4d_l$. Thus, in general, one might expect that d_c will depend on d_l. Recently, the growth of clusters in dimensions *above* d_c was studied by generating these clusters on a Cayley tree [21]. Finally, we note that our model with continuously tunable d_l serves to complement recent growth models for which d_f and d_g are continuously tunable [22].

We wish to thank F Leyvraz and I Majid for helpful discussion, W Marshall and the Boston University Academic Computing Center for time on the IBM 3081, and the NSF and ONR for financial support.

† Note that the effect of loops is to decrease the value of d_w to the value two as $d_l \to 2$.

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