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LETTER TO THE EDITOR

Probability densities for the displacement of random walks on percolation clusters

Shlomo Havlin†‡, David Movshovitz†, Benes Trus‡§ and George H Weiss‡

† Department of Physics, Bar-Ilan University, Ramat-Gan 52100, Israel
‡ Physical Sciences Laboratory and § Computer Systems Laboratory, Division of Computer Research and Technology, National Institutes of Health, Bethesda, MD 20205, USA

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Abstract. The probability density of the displacement or end-to-end distance of a random walk on the incipient infinite percolation cluster in $d = 2$ dimensions is studied by an exact enumeration method. Our numerical data suggest specific forms for the probability density both in the chemical distance variable l and the geometric distance r.

The problem of diffusion on fractals and other models for disordered and amorphous media is one of considerable current interest (Mandelbrot 1982, Alexander and Orbach 1982, Ben-Avraham and Havlin 1982, Gefen *et al* 1983, Pandey and Stauffer 1983, Havlin *et al* 1985a). Recently both Banavar and Willemson (1984) and O'Shaughnessy and Procaccia (1985) proposed a form for the probability density of displacement on a Sierpinski gasket and suggested that it had wider application to disordered fractal structures. In this letter we present data based on the exact enumeration method (Ben-Avraham and Havlin 1982, Majid *et al* 1984, Havlin *et al* 1984) that strongly support a different form for the probability density of displacement.

Two metrics have been found useful for the characterisation of distance on a fractal (Havlin 1984). One of these is obviously the geometric distance, r, and the second is the chemical distance, l, defined as the shortest distance between two points as measured along the structure (Havlin *et al* 1985b, and references cited there). The probability densities for displacement of a diffusing particle in these two metrics will be denoted by $P(r, t)$ and $\bar{P}(l, t)$ respectively, where, for example, $P(r, t)\,dr$ is the probability that a diffusing particle is at a geometric distance between r and $r + dr$ from its starting point, at time t.

In the present investigation an infinite percolation cluster at the critical concentration was generated on a two-dimensional rectangular lattice (Stauffer 1979, Alexandrowicz 1980). Statistical properties of the resulting random walk were obtained using the exact enumeration method (Havlin *et al* 1984). The exact enumeration method consists of setting a 0 at each lattice point belonging to the cluster and a 1 at the starting point of the random walk. The second step replaces the 0 at each cluster point neighbouring the origin by $1/n$, where n is the number of such points, and replaces the origin by a 0. Thus the random walker must move at each step. Successive steps repeat the procedure by dividing each non-zero entry equally among nearest-neighbouring points that belong to the cluster. At any step number the probability distribution

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for the location of the random walker is given by the numerical value at each lattice point. Thus, the method is exact for a random walk on any given cluster, but an average must be taken over a set of clusters to provide a reliable estimate of the probability distribution. Our results are based on averages taken over 1000 clusters with up to 2500 steps on a cluster.

We will assume that $P(r, t)$ and $\bar{P}(l, t)$ are normalised by

$$\int_0^\infty P(r, t)\,\mathrm{d}r = \int_0^\infty \bar{P}(l, t)\,\mathrm{d}l = 1. \tag{1}$$

The exponents d_w and d_w^l are defined by the relations

$$\langle r\rangle^{d_w} \sim t, \qquad \langle l\rangle^{d_w^l} \sim t \tag{2}$$

and d_l and d_f are the intrinsic and fractal dimensions, respectively (Vannimenus *et al* 1984, Havlin and Nossal 1984). The normalisation condition in (1) and the exponents in (2) allow us to suggest that $P(r, t)r$ and $\bar{P}(l, t)l$ can be written in terms of the scaled variables $x = r/t^{1/d_w}$, $y = l/t^{1/d_w^l}$ as

$$P(r, t)r \sim x^{d_f}F(x), \qquad \bar{P}(l, t)l \sim y^{d_l}G(y) \tag{3}$$

where $d_l/d_w^l = d_f/d_w = \bar{d}/2$ and $\bar{d}$ is the fracton dimension (Alexander and Orbach 1982). Our data were found to scale according to the relations in (3) for $5 \leqslant r \leqslant 150$, $10 \leqslant l \leqslant 100$, and $100 \leqslant t \leqslant 2500$ as can be seen from figures 1 and 2. The best scaling was obtained with the use of $d_w = 2.87 \pm 0.05$ and $d_w^l = 2.47 \pm 0.05$ in very good agreement with previous estimates of these exponents (Majid *et al* 1984). If we furthermore assume the following functional forms for $F(x)$ and $G(y)$

$$F(x) = \exp(-ax^\alpha) \qquad G(y) = \exp(-by^\beta) \tag{4a}$$

we obtain a best fit to the data with exponents

$$\alpha = 1.65 \pm 0.10, \qquad \beta = 1.90 \pm 0.10 \tag{4b}$$

where the limits on α, β, d_w and d_w^l were determined by varying the parameters and observing the quality of fit. The theoretical (full) curves in figure 1 were calculated using the *ansatz* of (4) in (3). It should be noted that the data for $P(r, t)$ are shown in averaged form. That is to say, each data point in figure 1 represents an average computed from five values of r that are close together. In contrast, the data points in figure 2 are raw data, suggesting that the chemical distance, l, is the more natural metric for the study of transport on percolation clusters.

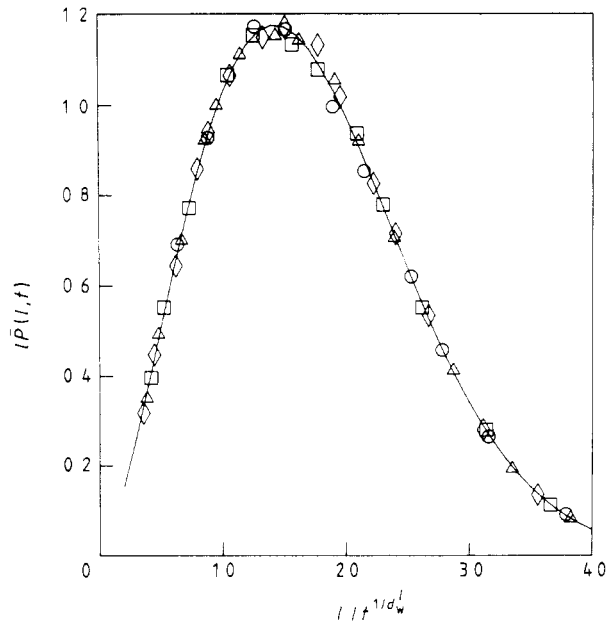
The forms for $P(r, t)$ and $\bar{P}(l, t)$ whose form and parameters have been found in the present investigation are consistent with the conditional density for the distribution of r given the value of l recently found by us (Havlin *et al* 1985b) which has the form

$$p(r\,|\,l) = Au^{\tilde{g}} \exp(-au^{\tilde{\delta}}), \qquad u = r/l^{\tilde{\nu}} \tag{5}$$

through the integral relation

$$P(r, t) = \int_0^\infty p(r\,|\,l)\bar{P}(l, t)\,\mathrm{d}l \tag{6}$$

which must be evaluated approximately, using an expansion around the maximum of the functions appearing in the exponent. The exponents appearing in (5) are $\tilde{g} = 2.5 \pm 0.3$, $\tilde{\delta} = 9.8 \pm 0.5$, $\tilde{\nu} = 0.88 \pm 0.02$. Substituting (5) and the function $\bar{P}(l, t)$ in (6)



13021 **Figure 1.** Plot of $l\bar{P}(l, t)$ against $l/t^{1/d_w^l}$ for different values of l and t. The different signs
13022 represent different number of steps $t = 1000$ (○), $t = 1500$ (□), $t = 2000$ (△), $t = 2500$ (◇).
13023 The range of l is $10 \leqslant l \leqslant 100$. The full curve represents the best fit to equations (3) and (4).

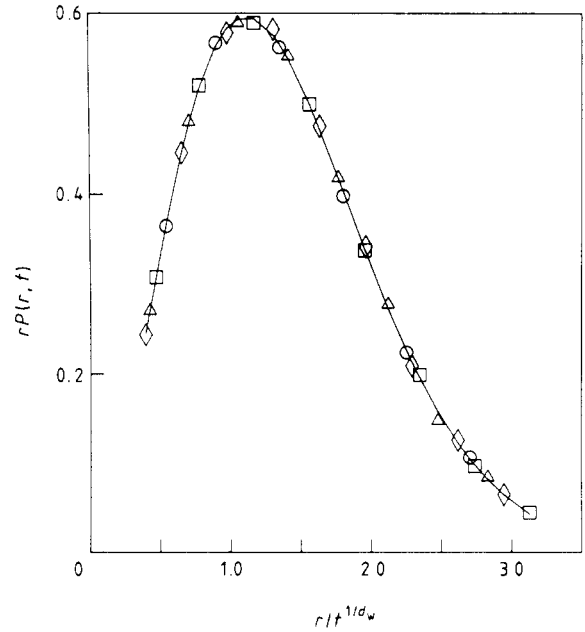


Figure 2. Plot of $rP(r, t)$ against $r/t^{1/d_w}$ for different values of R and t. The different signs represent different number of steps: $t = 1000$ (○), $t = 1500$ (□), $t = 2000$ (△), $t = 2500$ (◇). The range of r is $10 \leqslant r \leqslant 150$. The full curve represents the best fit to equations (3) and (4).

one obtains the following relation between α and β

$$\alpha = \beta\tilde{\delta}/(\beta + \tilde{\delta}\tilde{\nu}). \tag{7}$$

This relation is consistent with the values found numerically for α and β (Havlin *et al* 1985b). Very recently Banavar and Willemson (1984) and O'Shaughnessy and Procaccia (1985) suggested that the form of $P(r, t)$ for fractals is

$$P(r, t) \sim t^{-d_f/d_w} r^{d_f - 1} \exp(-ar^{d_w}/t). \tag{8}$$

Thus, for the case of percolation they predict $\alpha = d_w = 2.87$ (Majid *et al* 1984) which is quite different from our value $\alpha = 1.65 \pm 0.1$. The disagreement between our result and these theories does not seem to follow from the relatively short time range ($t \leqslant 2500$) for which the simulation was performed. The present good fit to a scaling function indicates rather that the results are already convergent to the appropriate $P(r, t)$ and $\bar{P}(l, t)$. We suspect, rather, that the theories leading to (8) did not take into account the effect of the random time spent by the random walker in exploring dead ends on the fractal. Indications that such effects lead to anomalies in the t dependence appearing in the exponent of (8) will be submitted shortly.

Very recently a different theory for the form of $P(r, t)$ was presented by Guyer (1985). In this theory the value of α is predicted to be $\alpha = d_w/(d_w - 1)$. For percolation it is found that $d_w = 2.87$ and therefore $\alpha \sim 1.53$ in agreement with our value of α.

The present approach (equations (3)-(7)) leads to an interesting prediction for the probability density of random walks on self-avoiding walks in two and four dimensions. Since for $d = 2$, $\tilde{\nu} = \nu_{saw} = \frac{3}{4}$, $\beta = 2$ and $\tilde{\delta}(1 - \tilde{\nu})^{-1} = 4$, it follows that $\alpha = \frac{8}{5}$, i.e. $p(r, t) \sim \exp(-ar^{8/5}/t^{3/5})$. For $d = 4$, $\tilde{\nu} = \frac{1}{2}$, $\beta = 2$ and $\tilde{\delta} = 2$. Thus $\alpha = \frac{4}{3}$, i.e. $p(r, t) \sim \exp(-ar^{4/3}/t^{1/3})$.

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