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LETTER TO THE EDITOR

Structure of clusters generated by random walks

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Abstract. We study the cluster structure resulting from a nearest-neighbour random walk embedded in a d-dimensional space. Each bond visited by the random walks is regarded as belonging to the cluster. The diffusion exponent and the fracton dimensional of the fractal cluster in $d=3$ is found to be $d_w = 3.5 \pm 0.1$ and $\bar{\bar{d}} = 0.57 \pm 0.02$, using a method of exact enumeration of random walks on these fractals.

The structure of statistical fractals is currently a subject of intensive study (Mandelbrot 1982, Stanley and Coniglio 1982, Meakin 1983, Alexander and Orbach 1982, Havlin and Nossal 1984). The main properties of interest are the fractal dimensionality of the clusters d_f, transport properties characterised by diffusion and resistivity exponents ($\bar{\zeta}$ and d_w, respectively) and the fracton dimensionality (Alexander and Orbach 1982) $\bar{\bar{d}} = 2d_f/d_w$. The possible relation between 'static' exponents like d_f and a 'dynamic' one like d_w is of special interest mostly because of the conjecture (Alexander and Orbach 1982, AO) relating these quantities for percolation clusters and the proposed extension (Meakin and Stanley 1983, Leyvraz and Stanley 1983) of this conjecture to other statistical fractals.

In this work we study the fundamental problem of the fractal structure resulting from a nearest-neighbour random walk (RW) of N_1 steps embedded in d-dimensional space. Each bond visited by the RW is regarded as a bond belonging to the cluster. The results for the limit $N_1 \to \infty$ in $d = 1$, 2 and 4 are obvious. For $d = 1$ and 2 the RW fills homogeneously all space and therefore one just obtains the homogeneous space with $\bar{\bar{d}} = d_f = d$. For $d = 4$ it is usually assumed that the RW intersects itself in a number of places which is negligible upon scaling for the purpose of calculating exponents (de Gennes 1979), thus the order of ramification is essentially 2 and the resulting fractal is quasi-linear with $d_f = 2$ and $\bar{\bar{d}} = 1$. However for the limit $N \to \infty$ in $d = 3$, the intrinsic structure of the fractal resulting from the RW has not so far been established. Thus it is of interest to study the diffusion exponents, d_w and $\bar{\bar{d}}$, of these fractals. We present scaling arguments as well as numerical data for d_w and $\bar{\bar{d}}$. Moreover, we relate the results found here for diffusion with the results of some recent work (Banavar *et al* 1983) on the resistivity of a random walk by the Einstein relation.

We study the diffusion exponent and the fracton dimensionality of a RW cluster by performing a second RW on it. The second RW whose starting point is uniformly distributed over the distinct sites visited by the first RW, is allowed to walk on any

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lattice bond visited by the first walker. Thus the second RW walks in a substrate which is ramified, in contrast to the situation treated in earlier analyses (Kehr and Kutner 1982). Assume that the fractal dimensionality of the RW substrate resulting from the first walker is d_f (R^{d_f} scales as the expected number of *distinct* sites visited where R is the linear span of the RW), and that the anomalous diffusion exponent of the second RW is d_w, then

$$\bar{\bar{d}} = 2d_f/d_w \tag{1}$$

is the fraction dimensionality (Alexander and Orbach 1982) of the RW substrate.

We seek the function $\langle R_2^2(N_1, N_2)\rangle$ giving the ened-to-end mean-square distance of an N_2-steps RW performed on a substrate resulting from an N_1-step RW. We first derive the exact result for the $d=1$ case. In this case

$$\langle R_2^2(N_1, N_2)\rangle = \int_0^\infty \langle R_2^2(L, N_2)\rangle g(L, N_1)\, \mathrm{d}L \tag{2}$$

where $g(L, N_1)$ is the probability density for the span L of the substrate random walk and $\langle R_2^2(L, N_2)\rangle$ is the mean square end-to-end distance of an N_2-step RW performed on the segment of length L. It can be shown that (Weiss and Rubin 1976)

$$g(L, N_1) \approx \frac{8}{(4\pi\sigma N_1)^{1/2}} \sum_{j=1}^{\infty} (-)^{j+1} j^2 \exp[-j^2L^2/(4\sigma N_1)] \tag{3}$$

where σ is the diffusion constant of the first walk. When the integral in equation (2) is evaluated, one finds

$$\langle R_2^2(N_1, N_2)\rangle = \tfrac{4}{3}(\sigma N_1)\ln 2 - \frac{128}{\pi^3}\sigma(N_1N_2)^{1/2} \times \left(\sum_{l=0}^{\infty} \frac{1}{(2l+1)^3} \frac{1}{1+\exp[\pi(2l+1)(N_2/N_1)^{1/2}]} + \frac{1}{\pi}\frac{1}{(2l+1)^4}\left(\frac{N_1}{N_2}\right)^{1/2} \ln\left(1+\exp\left[-\pi(2l+1)\left(\frac{N_2}{N_1}\right)^{1/2}\right]\right)\right). \tag{4}$$

We see that R_1^2 is of the form

$$\langle R_2^2(N_1, N_2)\rangle \approx N_1^a N_2^b f(N_2^c/N_1) \tag{5}$$

with $a=\frac{1}{2}$, $b=\frac{1}{2}$ and $c=1$. We assume this scaling form applies for any dimensionality d with suitable values of a, b, and c. In fact the limits of $f(x=N_2^c/N_1)$ for $x\to 0$ and $x\to\infty$ are easily obtained from simple arguments. For $x\to\infty$ the number of steps taken by the second walker is much larger than that taken by the first. Its span therefore limits $\langle R_2^2\rangle$ to be proportional to N_1 and independent of N_2. For $x\to 0$ the second walker is with overwhelming probability far from the edges of the fractal substrate resulting from the first RW, therefore $\langle R_2^2\rangle \approx N_2^{2/d_w}$ independent of N_1. Thus, we have

$$f(x) = \begin{cases} x^a & x\to 0 \\ x^{-b/c} & x\to\infty. \end{cases} \tag{6}$$

Substituting (6) into (5) yields

$$\langle R_2^2\rangle \approx \begin{cases} N_2^{b+ca} & N_1\to\infty,\ N_2 \text{ finite} \\ N_1^{(b+ca)/c} & N_1 \text{ finite},\ N_2\to\infty. \end{cases} \tag{7}$$

Our earlier remarks therefore imply the relations

$$c = 2/d_w, \qquad \tfrac{1}{2}bd_w + a = 1. \tag{8}$$

Introducing these identities into (5) one gets

$$\langle R_2^2\rangle/N_1 \approx (N_2^{2/d_w}/N_1)^{bd_w/2} f(N_2^{2/d_w}/N_1) \equiv g(N_2^{2/d_w}/N_1), \tag{9}$$

where

$$g(x) \approx \begin{cases} x & x \ll 1 \\ A = \text{constant} & x \gg 1. \end{cases} \tag{10}$$

Thus we see that only a single exponent, $2/d_w$, suffices to describe limiting properties of $\langle R_2^2\rangle$.

For the cases $d = 1$ and $d = 2$ we expect, for $N_1 \to \infty$, that $d_w = 2$ since the substrate of the first walk is the entire space. However, the limit $g(x) \approx x$, with $d_w = 2$, for $x \ll 1$ in (10) did not occur in our simulations in $d = 2$ even for $N_1 = 20\,000$ steps. The RW in $d = 3$ was studied carefully in the range $x \ll 1$ for which $R_2^{d_w} \approx N_2$ follows from (9) and (10). In figure 1 we present data for $d = 3$ and $x \ll 1$ obtained from calculating $d_w = \partial(\log N_2)/\partial(\log R_2)$. The data includes walks with $N_1 = 2000$, 3000, 4000, 5000 constituting the fractal RW substrates, on which $\langle R_2^2(N_1, N_2)\rangle$ was calculated by an exact enumeration procedure (Ben-Avraham and Havlin 1983, Magid *et al* 1984), up to $N_2 = 2000$. As seen from figure 1, the value d_w obtained is 3.5 ± 0.1. Since d_f of the RW substrate is 2 we obtain $\bar{\bar{d}} = 2d_f/d_w \simeq 4/3.5 \simeq 8/7$. This result contradicts the suggested generalisation of the conjecture of AO that $\bar{\bar{d}}$ should equal $\frac{4}{3}$ (Meakin and Stanley 1983, Leyvraz and Stanley 1983). This result for $\bar{\bar{d}}$ was also calculated directly, by analysing the probability of return to the origin from which it was found that $\bar{\bar{d}}/2 = 0.57 \pm 0.02$. The results for d_w and $\bar{\bar{d}}$ in different dimensions are summarised in table 1.

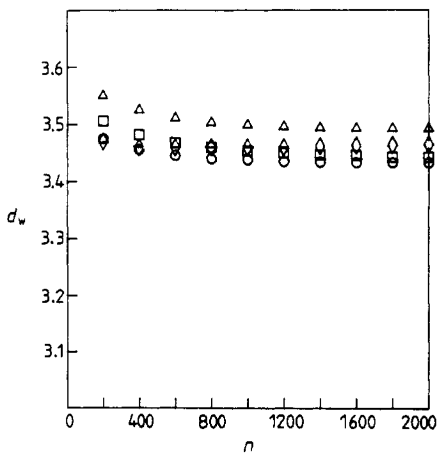

Figure 1. Results for d_w, the diffusion exponent of the second RW on the cluster generated by the first RW as a function of $n = N_2$ number of steps. The different symbols represent different values of N_1: ◇ for $N_1 = 2000$, △ for $N_1 = 3000$, ○ for $N_1 = 4000$ and □ for $N_1 = 5000$.

Table 1. Values for the exponents discussed in the text.

d	d_f	d_w	$\bar{\bar{d}}$	$\bar{\zeta}$
1	2	2	1	1
2	2	2	2	0
3	2	3.5 ± 0.1	$\frac{8}{7}$	1.5
>4	2	4	1	2

It is of interest to compare our results with a recent study (Banavar *et al* 1983) of the resistance between the end-to-end points of a RW. The resistance was found to be

$$r(N, d) \approx N^{x_1(d)} \tag{11}$$

where N is the number of steps, $x_1(d) = \frac{1}{2}$ for $d = 1$ and $= \frac{1}{4}d$ for $d = 2, 3, 4$ for the case when overlapping bonds are regarded as a single resistor. In order to relate our results to (11) we use the relation (Alexander and Orbach 1982)

$$d_w = d_f + \bar{\zeta} \tag{12}$$

where $\bar{\zeta}$ is the resistivity exponent defined by $r \approx r^{\bar{\zeta}}$ where r is the resistance between two bars of size R separated by a distance R.

$$r(N, d) \approx R^{2x_1(d)} \approx R^{\bar{\zeta}(d)} \tag{13}$$

with $\bar{\zeta} = 1, 1, 1.5, 2$ for $d = 1, 2, 3, 4$ respectively. These results for $\bar{\zeta}$ (or $x_1(d)$) can be obtained by using (12). For $d = 1$ it is obvious that $d_f = 1$, $d_w = 2$, $\bar{\zeta} = 1$. For $d = 4$ as mentioned earlier $d_f = 2$, $d_w = 4$, so that $\bar{\zeta} = 2$. The case of $d = 3$ is not trivial. In this case $d_f = 2$ and our result $d_w \cong 3.5$ implies that $\bar{\zeta} = 1.5$. The case of $d = 2$ is not consistent with (12) since we argue that the first walk occupies the entire space, $d_w = 2$, $d_f = 2$ thus $\zeta \to 0$ in contrast to $\bar{\zeta} = 1$ obtained by Banavar *et al* (1983). This discrepancy may be explained by the fact that we study the case of $N_1 \to \infty$ and Banavar *et al* (1983) the case of finite N_1. However, our general scaling assumptions (5) and (9) do not predict another exponent for this case ($N_1 \approx N_2$), it is rather represented as a crossover regime, (10).

To conclude, we have dealt with the problem of diffusion on a RW substrate. We solve exactly the one-dimensional case which suggests the form of the scaling function for all d. For the non-trivial case of $d = 3$ we find $d_w \approx 3.5$ and $\bar{\bar{d}} \approx \frac{8}{7}$. These results are consistent with recent results on the resistivity for the RW substrate.

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