PHYSICAL REVIEW E, VOLUME 63, 010102(R)

Universality of the directed polymer model

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(Received 18 September 2000; published 27 December 2000)

The universality of the directed polymer model and the analogous Kardar-Parisi-Zhang equation is supported by numerical simulations using non-Gaussian random probability distributions in two, three, and four dimensions. It is shown that although in the non-Gaussian cases the *finite size* estimates of the energy exponents are below the presumed universal values, these estimates *increase* with the system size, and the further they are below the universal values, the higher is their rate of increase. The results are explained in terms of the efficiency of variance reduction during the optimization process.

DOI: 10.1103/PhysRevE.63.010102 PACS number(s): 05.40.−a

The directed polymer model [1] and the analogous Kardar-Parisi-Zhang (KPZ) equation [2] have drawn significant attention recently [3]. The original introduction of the model [1] considered directed polymers in random media, and more recently it has been applied to model other processes such as tearing and cracks formation [4]. The study of the model is focused on the characteristics of the optimal path which connects two distant points in a random environment. Passing through any point of that environment is associated with a random valued cost, usually expressed in units of energy or time. Unlike the simple case of constant cost, the path of minimal cost (the optimal path) is usually not the straight line which connects the two points, and its mean transversal distance from the straight line which connects its endpoints grows as a power law with the distance between the endpoints. Another power law behavior is associated with the relation between the variance of the energy cost (or time cost), and the distance between the endpoints. These two power laws are expressed in terms of their exponents, which are connected by a scaling relation. The random environment is usually simulated by a lattice whose bonds (or sites) are assigned with random values, and it is generally believed that the characteristics of the optimal path are universal, i.e., apart from very special cases, the exponents depend only on the dimension of the lattice and not on the details of the lattice structure, or the type of randomness associated with it.

The universality hypothesis was recently challenged by numerical results [5] which showed that for dimensions higher than 2, the values of the estimated exponents do depend on the type of the probability distribution which is associated with the random lattice. In particular, non-Gaussian distributions yield lower values for the exponents compared to Gaussian distribution values, which are the presumed universal values. This publication has raised strong opposition in the form of three recent articles [6–8], which argued that the estimates presented in [5] are affected by finite size effects, and thus they are not the asymptotic values of the exponents in these cases. Two of these articles [6,7] independently suggested that the estimates presented in [5] are disturbed by directed percolation effects. However, none of these opposing articles presented any alternative numerical estimates to those presented in [5].

This Rapid Communication presents results of numerical simulations for the three- and four-dimensional cases studied in [5], but for larger lattices. The results show that the estimates presented in [5] are indeed below the asymptotic values of the exponents. Moreover, it is shown that though convergence to asymptotic values happens at lattice sizes much bigger than studied, the data suggest the presence of the universal values in all cases. In the second part of the article the directed percolation explanation presented in Refs. [6,7] is challenged, and an alternative explanation is presented in terms of the efficiency of variance reduction during the optimization process.

In two dimensions, the directed polymer model can be described as follows: In a square lattice which was cut along its diagonal and oriented as a triangle whose apex is up and the diagonal is its base, we assign to each bond or site a random number. Of all the paths leading from the apex (the origin) to the base, we refer only to those whose direction is *always* down to the base. For each one of these paths we calculate the sum of the random numbers along it, which defines its value. We focus our interest on the path of minimal value which we call ''the optimal path,'' and define the space exponent ν by $D \sim t^{\nu}$, where D is the mean distance of the endpoint of the optimal path from the center of the base, and t is the size of the triangle. In terms of a polymer which orients itself along the optimal path, t is the length of the directed polymer chain. Another exponent which characterizes this model is the energy exponent ω, defined by $\sigma_E \sim t^{\omega}$, where E is the value of the optimal path (or the energy of the directed polymer), and σ_E is its standard deviation. These two exponents are related by the scaling relation $\omega = 2\nu - 1$ [9]. In the two-dimensional case, the values of the exponents are $\nu \simeq 2/3$, $\omega \simeq 1/3$, and the values 2/3 and 1/3 are considered to be exact. In dimensions higher than 2, there are only numerical estimates for the values of the exponents, and in the three- and four-dimensional cases, the accepted values for the exponents are $\nu \simeq 0.62$, $\omega \simeq 0.24$, and $\nu \simeq 0.59$, $\omega \simeq 0.18$ respectively [10,11]. However, these numerical estimates were obtained for random values taken from Gaussian distribution, while non-Gaussian distributions, though preserving the space exponents [11], yield lower estimates for the energy exponents [5,11]. The dependence of the estimated energy exponents on the form of the probability dis-

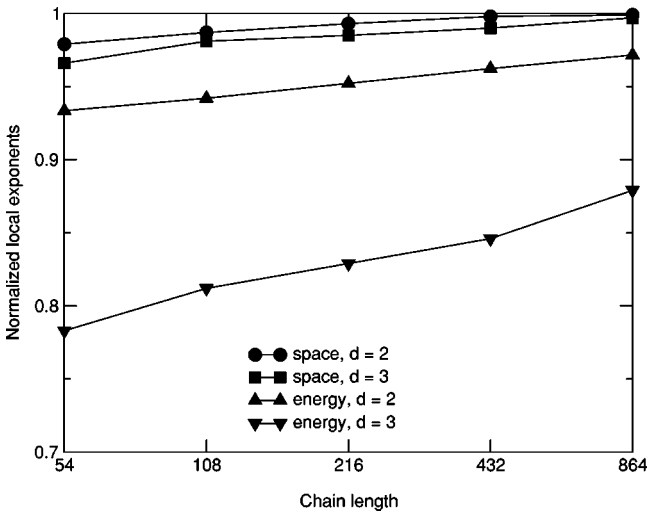

FIG. 1. Local exponents of ν and ω (normalized to the universal values), as a function of the chain length for $d=2$ and $d=3$. The random bonds values are taken from uniform distribution.

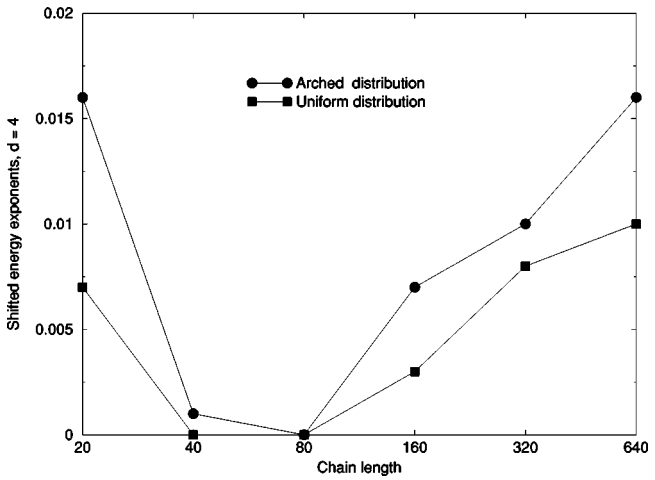

FIG. 2. Local exponents of ω as a function of the chain length for the random site lattice in $d=4$. The results are shifted by their minimal values: 0.116 for the uniform distribution, and 0.048 for the arched distribution, thus presenting the difference between the local exponents and their minima.

tribution suggests nonuniversality of the model [5], and a break of the scaling relation. It is this discrepancy which is addressed in our study.

Our numerical simulations for $d=2$ and $d=3$ were performed with random bonds whose values were taken from uniform distribution. The results are presented in terms of local exponents, computed by $\log_4[V(2t)/V(t/2)]$, where V is either D or σ_E. Since we are interested in the relation between the local exponents and the universal values, the results are presented in Fig. 1 by normalized local exponents, where the estimated values are divided by the presumed universal values: $\nu=2/3$, $\omega=1/3$, in two dimensions, and $\nu=0.62$, $\omega=0.24$, in three dimensions. The results of Fig. 1 can be summarized as follows:

(1) All local exponents *steadily increase* with t.

(2) In both two and three dimensions the local space exponents have practically reached their universal values.

(3) The local energy exponents in the three-dimensional case are lower than those of the two-dimensional one, but their rate of increase is higher.

The results related to the two-dimensional local energy exponents supply a test for the validity of a recent suggestion proposed in [8]. According to this suggestion, the difference between the local exponents and their asymptotic values decreases in proportion to $t^{-\gamma}$, where $\gamma=0.23\pm0.02$. However, the data presented in Fig. 1 suggests that the convergence towards the asymptotic value in this case (and in this range of chain lengths) is characterized by logarithmic rather than power law dependence on the chain length. The probability of power law dependence, which is determined by the measurement errors, was calculated and found to be nil.

The numerical simulations of the four-dimensional case were performed on the random site version of the model [5], with two types of probability distributions: the uniform distribution, and the arched distribution: $p(r)\sim(1-|r|)^{-1/2}$, $-1<r<1$. In both cases the local space exponents approach their Gaussian values of $\simeq 0.59$, but in both cases the local energy exponents are much lower than their Gaussian values. Moreover, the local energy exponents initially decline, and the crucial question is whether, as suggested in [5], their minimal values are also their asymptotic values. Thus, in Fig. 2 the estimates of the local energy exponents are shifted by their minimal values: 0.116 in the uniform distribution case, and 0.048 in the arched distribution case, showing the difference between the estimates and their minima. As can be clearly seen, after the fall comes the rise, and the local exponents of the arched distribution case, which are lower than the local exponents of the uniform case, also increase faster, similar to the picture presented in Fig. 1.

In conclusion, Figs. 1 and 2 present a coherent picture of systems which have not yet reached their asymptotic local exponents, and whose local energy exponents are much more sensitive to the form of the probability distribution than their local space exponents. The local space exponents are close to the Gaussian values, while the further is a local energy exponent below the Gaussian value, the higher is its rate of increase. These results provide in our opinion a solid support for the universality hypothesis.

As mentioned above, both Refs. [6] and [7] suggested an explanation of the results of [5] in terms of directed percolation effects. In the directed percolation model, the random bonds (or sites) are assigned with values taken from the bimodal (0,1) distribution, with probability p to get zero valued bonds. The set of sites which are connected by zero valued bonds to the origin compose a directed percolation cluster. In the case of $p>p_c$, where p_c is called the critical probability, there is a finite probability to get clusters of infinite length. From the directed polymer model point of view, in the case of bimodal (0,1) distribution and $p>p_c$, after an initial search of random duration, the optimal path orients itself along a directed percolation cluster, and thus the mean en-

ergy and energy variance of the optimal paths reach constant values which are independent of system size, and the energy exponent is zero valued.

In [6] it was shown that in the case of $p<p_c$, the pattern of the local energy exponents is similar to those found in the worst cases studied in [5]. Both [6] and [7] address the fact that the higher is the low end of the random site continuous distribution, and thus the more it resembles the low end of the bimodal distribution, the lower are the local energy exponents. Both of them also argue that the much lower than universal values recorded in [5] for higher dimensions, are the result of the lower p_c of directed percolation in higher dimensions. A lower p_c implies that the same continuous distributions resemble more faithfully the critical bimodal distribution with its zero valued energy exponent.

In their reply to the comment of [7], Newman and Swift [12] have already challenged the directed percolation explanation, and argued that results obtained for the uniform distribution are unlikely to be an outcome of bimodal distribution (or directed percolation) effects. To this critique we would like to add another: The crux of the matter in the bimodal distribution case, both above and below p_c, is the ability of the optimal paths to orient themselves along directed percolation clusters without any change in their energy values [13]. No such situation can occur in the continuous distribution case. This fundamental difference is the reason for another difference between the bimodal and continuous cases: In the bimodal case, not only the energy variance of the optimal paths grows slowly with lattice size, but so does also their mean energy, whose growth rate approaches logarithmic dependence on lattice size as p approaches p_c. Actually, the slow growth rate of the mean energy, combined with the fact that no optimal path can decrease its energy, lead inevitably to the slow growth rate of the energy variance in the bimodal case. In contrast, in all the continuous cases studied in [5], the mean energy is almost proportional to the lattice size. The difference between logarithmic dependence and linear dependence is a matter of substance, not of a degree.

Moreover, the whole discussion until now focused on the cases of lower than universal values of the local exponents. But opposite cases also exist. Figure 3 presents local exponents obtained for a simple two-dimensional lattice whose bonds are assigned with the negative values of random numbers taken from the Gaussian distribution and raised to powers of $k=2,4,6$. The probability distribution in these cases is $p(r)\sim|r|^{-(k-1)/k}e^{-1/2|r|^{2/k}}$; $r<0$, and they all have massive left tails whose length grows with k. As can be seen in Fig. 3, a longer left tail is associated with higher local exponents, which are even higher than 1/2 at some stages.

Since the optimal paths orient themselves along lower than average energy bonds, the important part of the bond probability distribution is its left-hand side (LHS). The numerical evidence which should be explained can be summarized as follows: A decreasing (right tailed) LHS yields initially lower than universal local exponents. An increasing (left tailed) LHS might yield initially either lower or higher than universal values; the longer the left tail, the higher the local exponents.

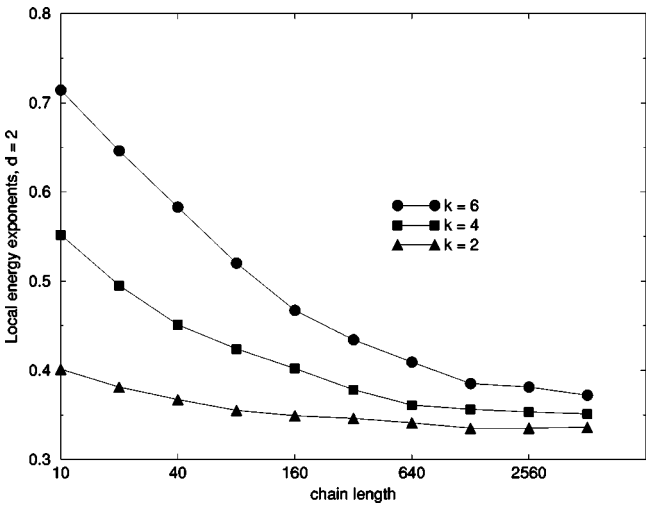

FIG. 3. Local exponents of ω as a function of the chain length. The results are shown for three random bond probability distributions in $d=2$. The bonds are assigned with the negative values of random numbers taken from the Gaussian distribution and raised to powers of $k=2,4,6$.

In the following, an alternative to the directed percolation explanation is presented. This alternative explanation relies heavily on the form of the optimal paths energy distribution, which for long enough paths arrives eventually at a fixed functional form. The presumed universality of the model is connected with the assumption that apart from very special cases, such as the bimodal distribution at $p>p_c$, this functional form is independent of the initial probability distribution of the random bonds. The asymptotic functional form resembles a Gaussian whose left tail is longer than its right tail, and the higher the dimension, the more skewed to the left is the form.

Two different mechanisms determine this form of the asymptotic probability distributions: The first is the growth of the system, which causes the energy of any random path to have eventually a Gaussian distribution, regardless of the random bond distribution. The second mechanism is the optimization process, which favors lower values, and thus transfers weight from the right side to the left side of the optimal paths energy distribution.

Let us consider a right tailed optimal paths energy distribution, and a left tailed one. In both cases a major part of the variance comes from the tail. When the optimization process is performed on the right tailed distribution, it transfers probabilistic weight from the tail to the center, thus reducing effectively the variance of the distribution. In contrast, when the optimization process is performed on the left tailed distribution, it transfers weight from the center to the left tail, thus increasing the variance. In the case of a symmetric distribution, weight is deleted from the right tail and added to the left tail, but since the center also moves to the left, the variance is reduced. Actually, even in many cases of pure left tailed distributions, the movement of the center to the left more than compensates for the additional weight at the left tail, and the result is a decrease in the variance.

As a result of the optimization process, the asymptotic

probability distribution is skewed to the left, the rate of growth of the variance is lower than 1, and the mean energy is not strictly proportional to lattice size. Of course, the higher the dimension, the more nearest neighbors each site has, and the more efficient the optimization-variance reduction process. This is the reason for both the more skewed to the left asymptotic probability distribution, and the lower values of the exponents in higher dimensions.

Moreover, if in the intermediate stage the optimal paths distribution is less skewed to the left than its asymptotic form, the variance reduction efficiency is higher than in the final stage, and thus the local exponents are lower than their asymptotic (universal) values. The opposite happens in the case of an intermediate distribution which is more skewed to the left than the asymptotic form. It should only be added that the form of the optimal path probability distribution resembles initially the form of the random bond probability distribution, and this analysis explains all the numerical evidence summarized above.

A recent article [14] discusses the random bond probability distribution which would lead to the fastest convergence of the local energy exponents to their asymptotic values. According to the above analysis, the best choice is simply a random bond distribution which resembles the asymptotic form of the optimal paths distribution.

The last issue which should be considered is the larger deviations from universal values recorded for the same random site distributions in higher dimensions. According to the above analysis, the reason for these larger deviations is not the smaller difference between these distributions and the critical bimodal distribution, but the bigger difference between these distributions and the more skewed to the left asymptotic form of the optimal paths probability distribution.

In conclusion, this study supports the universality of the directed polymer model in two ways. First, it is shown that finite size effects do influence the estimated local exponents of the non-Gaussian distributions in higher dimensions. Then, these finite size effects are explained by the form of the optimal paths probability distribution, and its influence on the efficiency of variance reduction during the optimization process.

We wish to thank the German Israel Foundation (GIF) for financial support.

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