Optimal Path in Strong Disorder and Shortest Path in Invasion Percolation with Trapping

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We present analytical and numerical results suggesting that the optimal path in an energy landscape in the strong disorder limit is in the universality class of the shortest path in invasion percolation with trapping. Our results imply that, in contrast to common belief, invasion percolation with trapping and regular percolation in $d = 3$ are in different universality classes. [S0031-9007(97)04589-4]

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The question how to find and to characterize the optimal path in a disordered energy landscape has been the subject of much recent interest. Whenever the energy depends on many variables of the considered disordered system the energy landscape is rugged and optimization is crucial to find its global minimum. Prominent examples include spin glasses [1], folding of proteins [2], and the well-known traveling salesman problem [3]. For an energy landscape on a directed lattice both an algorithm for finding the optimal path and analytical results for the geometry of the path exist [4,5].

In contrast, in the nondirected case only very little is known. Recently, Cieplak *et al.* [6] developed a novel algorithm to study the optimal path in the strong disorder limit, where the energies ε_i associated with the bonds of a given, fully occupied lattice are taken from a very broad distribution, such that the energy E_{AB} for a path between two points A and B is given by the maximum energy barrier along that path. In this case, the optimal, i.e., minimum energy path can be constructed by ranking the energy values and removing the bonds according to their rank, starting with the highest energy value. There is one constraint: If the removal of a bond will break the connection between the two opposite faces of the lattice, the bond is not removed and one continues with the next bond in the rank list. This procedure is continued until a single path remains. Since the energies are *randomly* assigned to the bonds, an *equivalent* procedure is just to remove bonds randomly with the constraint stated above. Cieplak *et al.* find that the optimal path constructed by this method is a fractal. The length ℓ of the path scales with the end-to-end distance r as $\ell \sim r^{d_{\rm opt}}$ with $d_{\rm opt} = 1.22 \pm 0.01$ in $d = 2$, $d_{\rm opt} = 1.42 \pm 0.02$ in $d = 3$, and $d_{\rm opt} = 1.59 \pm 0.02$ in $d = 4$ [6,7], which is different from the scaling of the shortest path in regular percolation [8]. Moreover, they introduced a new variant of invasion percolation (compressible invasion percolation, see below, where loops are forbidden to occur), and showed that the shortest path in this model is in the same universality class as the optimal path in the strong disorder limit [7].

Invasion percolation (IP) has been introduced by Wilkinson and Willemsen [9] as a model to describe the evolution of the interface between an invading fluid and a "defending" fluid in a porous medium. Apart from this application, interest in IP arises from the fact that it is parameter free and self-organizes into a critical state [10]. One commonly distinguishes between two types of invasion percolation. (i) In the most common type of IP, the defending liquid is considered as incompressible, and is trapped in the medium when surrounded by the invader. The invader cannot penetrate into a trapped regime. This system is called trapping invasion percolation (trapping IP). (ii) The defending liquid is fully compressible, and therefore the invader can enter also a trapped regime. We will refer to this as compressible invasion percolation (compressible IP). While compressible IP is believed to be in the same universality class as regular percolation for all dimensions d, the trapping IP was found to belong to a different universality class in $d = 2$. It is believed that for $d \geq 3$, trapped regimes, where the defender is fully surrounded by the invader, are irrelevant and trapping IP belongs to the universality class of regular percolation [9].

In this Letter we show that the shortest paths in trapping IP and in the loopless compressible IP introduced by Cieplak *et al.* [6] are in the same universality class for all dimensions d. This finding has two important consequences: (1) the optimal path in the strong disorder limit is in the same universality class as the shortest path in trapping IP, and (2) in contrast to common belief, trapping IP and regular percolation are *not* in the same universality class for $d = 3$.

To show that the shortest path in loopless compressible IP belongs to the universality class of the shortest path in trapping IP we consider a mapping which is valid for all d. First, we define the trapping IP on a bond lattice. We consider a square lattice in $d = 2$ where each bond is regarded as a pipe connected to six neighboring pipes. In Fig. 1 we show an example of three first steps of an invasion process. When the invader penetrates a bond, it also fills its two neighboring crossings. Thus, as shown

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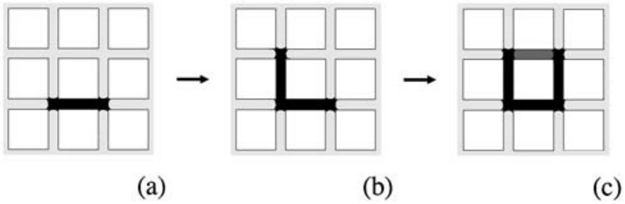

FIG. 1. Illustration of the three first steps of an example of an invasion process. In the third step (c) a trapped regime occurs. The invader is in black, the defender is in light grey, and the trapped region is in dark grey.

in Fig. 1(c), a bond filled with defender and surrounded on both sides by invader is trapped. This definition can be easily generalized to any d-dimensional bond lattice.

For illustration, we first describe the mapping for $d = 2$. We consider two lattices with identical random numbers assigned to each bond. On the first lattice, we generate a trapping IP cluster [Figs. 2(a), 2(c), and 2(e)], and on the second lattice we generate a loopless compressible IP cluster [Figs. 2(b), 2(d), and 2(f)]; both clusters start from the same seed. Figures 2(a) and 2(b) show the clusters just before the first trapped regime occurs in trapping IP, where both growth processes generated *identical* clusters. In the next step, the bond creating the trapped region in trapping IP is invaded, creating in the first lattice [Fig. 2(c)] a trapped region that cannot be invaded further [dark grey bonds in Fig. 2(c)], and having no further effect in the second lattice [Fig. 2(d)]. In the following steps, the clusters remain identical outside that regime [in Figs. 2(e) and 2(f)]; only inside they can be different. Nevertheless, in both cases no shorter path can occur leading from one side of this regime to the other side.

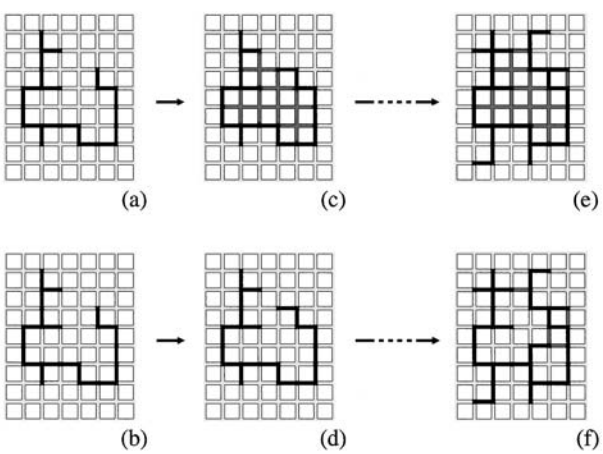

FIG. 2. Illustration of the mapping between the shortest path in trapping IP [(a), (c), and (e)] and loopless compressible IP [(b), (d), and (f)] for a bond lattice in $d = 2$. The figure shows an example of two clusters grown on a lattice with identical random numbers starting from the same seed, (a),(b) just before and (c),(d) just after the step in which the first trapped regime in trapping IP occurs, as well as (e),(f) few steps later. The invader is in black, the defender is in light grey, and the trapped defender region as well as the blocked bonds are in dark grey.

These arguments can be easily extended to bond lattices in $d > 2$. Also here, the clusters created on a lattice with identical random numbers differ only inside regimes which are trapped in trapping IP. Nevertheless, as in $d = 2$, no shorter path crossing these regimes can occur in both cases. For demonstration a simple example of a trapped regime in $d = 3$ is shown in Fig. 3. Therefore we can argue that for all dimensions d the shortest paths in trapping IP and in loopless compressible IP are in the same universality class. Since that one in loopless compressible IP was shown to be in the same universality class as the optimal path [7] it implies that the optimal path in the strong disorder limit is in the same universality class as the shortest path in trapping IP.

In order to test our conclusions, we have performed numerical calculations to determine the fractal dimension $d_{\min}$ of the shortest paths (i) in trapping IP [see Fig. 4(a)] and (ii) in loopless compressible IP [see Figs. 4(b) and 4(c)]. The fractal dimension $d_{\min}$ is defined in analogy to d_{opt} and describes how the length ℓ of the path between two points scales with their Euclidian distance r, $\ell \sim r^{d_{\min}}$. For loopless compressible IP, $d_{\min}$ has been studied before [7]. For simplicity, we restrict ourselves in case (i) to site lattices, and in (ii) we study both site and bond lattices. We also study the fractal dimension d_{opt} of the optimal path in the limit of strong disorder [see Fig. 4(d)]. For constructing the optimal path we employ a modified algorithm [11], which is based on removing bonds of the percolation backbone at criticality and enables us to consider larger systems. The similar results for $d_{\min}$ and d_{opt} clearly support our conclusion that all three paths, the shortest path in trapping IP, the shortest path in loopless compressible IP, and the optimal path in the limit of strong disorder, are in the same universality class. Since the numerical results for trapping IP and loopless compressible IP are obtained for site lattices it further supports the universality conclusion.

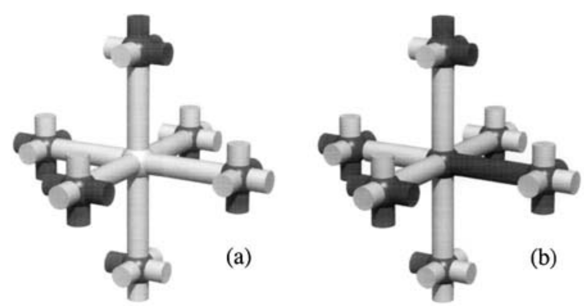

FIG. 3. Illustration of a simple trapped regime in $d = 3$; (a) in trapping IP this regime cannot be penetrated, and (b) in loopless compressible IP the regime can be penetrated, as shown. Nevertheless, in both cases a shorter path leading from one side of the regime to the other side cannot occur. The invader is in dark grey and the defender is in light grey.

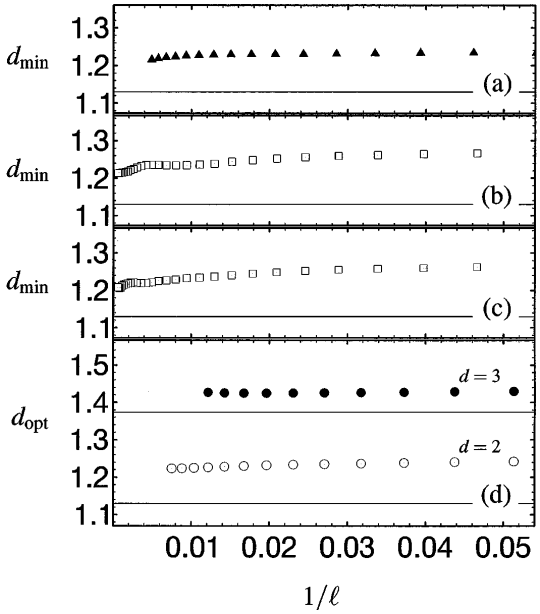

FIG. 4. Plot of (a)–(c) the fractal dimension of the shortest path, $d_{\rm min}$, in $d = 2$ and of (d) the optimal path in the strong disorder limit, $d_{\rm opt}$, in $d = 2$ and $d = 3$, as a function of $1/\ell$ [obtained from successive slopes of $\ln r(\ell)$ vs $\ln \ell$]. The data are obtained for (a) trapping IP on a square bond lattice, (b) loopless compressible IP on a square site lattice, (c) loopless compressible IP on a square bond lattice, and (d) optimal path in the strong disorder limit on a square ($d = 2$) and simple cubic ($d = 3$) bond lattice. For (a) more than 6×10^4 systems of linear size $L = 501$ (full triangle), and for (b),(c) more than 10^4 systems of linear size $L = 4001$ (open square) are used. For (d) for both $d = 2$ (open circle) and $d = 3$ (full circle) ensembles of more than 2.5×10^5 paths are used. In all cases, the error bars are about the size of the symbols. The horizontal lines indicate the values of the fractal dimensions $d_{\rm min} = 1.130 \pm 0.004$ in $d = 2$ [13] and $d_{\rm min} = 1.374 \pm 0.004$ in $d = 3$ [13] of the shortest path in regular percolation.

Our finding can be used to shed light on the important question as to which universality class trapping IP in three-dimensional space belongs. This question could, so far, not be decided by numerical calculations, since the trapping constraint in $d \geq 3$ is very hard to simulate. It is, however, commonly believed that trapping IP in $d \geq 3$ belongs to the same universality class as regular percolation, because trapped regimes, where the defender is fully surrounded by the invader, are rare and therefore irrelevant. Since we show that the optimal path belongs to the same universality class as the shortest path in trapping IP for all d, and since its fractal dimension $d_{\rm opt} = 1.43 \pm 0.01$ [see Fig. 4(d) and [7]] is different from the fractal dimension $d_{\rm min} = 1.374 \pm 0.004$ [13] of the shortest path in regular percolation, we can conclude that trapping IP is *not* in the universality class of regular percolation. Of course, for higher dimensions the difference becomes less and less pronounced; already for $d = 4$ the exponent obtained for the optimal path $d_{\rm opt} = 1.59 \pm 0.02$ [7] is nearly indistinguishable from the exponent for regular percolation $d_{\rm min} = 1.61 \pm 0.02$ [13]. However, since the optimal path cannot be shorter than the minimal path it follows that $d_{\rm min} \leq d_{\rm opt}$ for all d.

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