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Directed-polymer and ballistic-deposition growth with correlated noise

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We present numerical studies of the effect of long-range correlated noise on (i) the nonlinear Kardar, Parisi, and Zhang (KPZ) stochastic differential equation and the related problem of directed-polymer (DP) growth, and (ii) the ballistic-deposition (BD) model. The results for the KPZ and DP models are consistent with each other, and agree better with one recent theoretical prediction of Hentschel and Family [Phys. Rev. Lett. **66**, 1982 (1991)] than with other theoretical predictions. Contrary to the general belief that BD is described by the KPZ equation, we find the surprising result that BD with correlated noise belongs to a *different universality class* than the KPZ equation.

Disorderly surface growth has been receiving much attention recently [1–4]. Many different models—such as ballistic deposition (BD), Eden growth, solid-on-solid deposition—are believed to belong to the same universality class, which is described by the Kardar, Parisi, and Zhang (KPZ) stochastic differential equation [5]

$$\frac{\partial h}{\partial t}=\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(\mathbf{x},t)\,. \tag{1}$$

Here $h(\mathbf{x},t)$ is the deviation from the mean of the height of the surface and η is the stochastic variable (noise) describing local variations in growth rate. One justification of the KPZ equation arises from its predicted scaling form for height-height correlation function

$$\langle|h(\mathbf{x},t)-h(\mathbf{x}',t')|^2\rangle^{1/2}\sim|\mathbf{x}-\mathbf{x}'|^{\alpha}f\left(\frac{|t-t'|}{|\mathbf{x}-\mathbf{x}'|^{z}}\right), \tag{2a}$$

which is consistent with the scaling relation observed in various surface growth models [6],

$$w(L,t)\equiv\langle h^2(\mathbf{x},t)\rangle^{1/2}\sim L^{\alpha}f\left(\frac{t}{L^{z}}\right). \tag{2b}$$

Here the average is taken over all positions $\mathbf{x}$ in the system of finite size L and $w(L,t)$ is the width of the surface at time t. For $1\ll t\ll t_x$, $w(L,t)\sim t^{\beta}$, where $t_x\sim L^z$ and $\beta=\alpha/z$. For $t\gg t_x$, $w(L,t)\sim L^{\alpha}$. Numerical studies of the roughening exponents for various surface growth models in $1+1$ dimensions (one-dimensional surface) with random noise agree well with the values predicted by the KPZ equation ($\alpha=\frac{1}{2},\beta=\frac{1}{3}$).

When the noise itself is the result of another stochastic process, then the noise *cannot be treated as random*—the noise is correlated in space and/or time [7]. In this case, the exponents depend on the strength of the correlation. Medina *et al.* [8] used dynamical renormalization-group analysis to study the KPZ equation with long-range correlated noise. The noise they studied has the correlation

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle\sim|\mathbf{x}-\mathbf{x}'|^{2\rho-d}|t-t'|^{2\theta-1}, \tag{3a}$$

where $d+1$ is the dimension of the system (d is the dimension of the surface). If the noise has no temporal correlation, i.e.,

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle\sim|\mathbf{x}-\mathbf{x}'|^{2\rho-d}\delta(t-t'), \tag{3b}$$

the exponents obey the relation $\alpha+z=2$. Since then there is only one independent scaling exponent, it is sufficient to give β; for $d=1$

$$\beta=\begin{cases}1/3, & 0<\rho\le\frac{1}{4}\,,\\ (1+2\rho)/(5-2\rho), & \frac{1}{4}<\rho\le 1.\end{cases} \tag{4a}$$

The other feature of the KPZ equation is that it can be mapped to the directed-polymer (DP) problem [9]. The noise plays the role of a time-dependent random potential. Thus, the results of Ref. [8] can also apply to the DP problem in a correlated potential field.

Zhang [10] used a replica method to study the DP problem with correlated noise η given by Eq. (3b). Due to the analogy between the DP problem and the KPZ equation, Zhang predicts for $d=1$

$$\beta=\begin{cases}(1+2\rho)/(3+2\rho), & 0<\rho\le\frac{1}{2}\,,\\ (1+2\rho)/(5-2\rho), & \frac{1}{2}<\rho\le 1.\end{cases} \tag{4b}$$

Very recently, Hentschel and Family [11] studied the scaling behavior for dissipative dynamical systems and proposed a new relation:

$$\beta=1/(3-2\rho),\ 0\le\rho\le\tfrac{1}{2}\ . \tag{4c}$$

Note that the three predictions [Eqs. (4a), (4b), and (4c)] differ for $0<\rho<\frac{1}{2}$.

There have been several prior attempts to verify the analytical results with correlated noise [12]. This work relies on numerical methods that probably generate undesired correlations in the noise. Here we generate algebraically correlated noise [13], integrate numerically the KPZ equation, and also simulate the DP growth in a correlated potential field. Our results for KPZ and DP agree with each other, and qualitatively agree somewhat better with (4c) than with (4a) or (4b). Finally, we implement correlated noise into the BD model, and were surprised to find surface roughening exponents that differ from *both* the KPZ equation *and* the DP problem.

To construct the algebraically correlated noise, we first generate a representation of random Gaussian uncorrelated noise η_0 $(\mathbf{x},t)$, then Fourier transform it to obtain $\eta_0(\mathbf{q},\omega)$. We define

$$\eta(\mathbf{q},\omega)\equiv|\mathbf{q}|^{-\rho}|\omega|^{-\theta}\eta_0(\mathbf{q},\omega)\ . \tag{5}$$

The noise $\eta(\mathbf{x},t)$ is obtained by Fourier transforming $\eta(\mathbf{q},\omega)$ back into the space and time domain. It is straightforward to verify that $\eta(\mathbf{x},t)$ obtained in this way has the correct correlations (3a). We restrict ourselves to the $d=1$ case and the noise has only spatial correlation ($\theta=0$) as in Eq. (3b) [14].

(i) Consider first the KPZ equation with noise η described by (3b). For a one-dimensional surface, the discrete form of Eq. (1) is

$$h_{t+\Delta t}(i)=h_t(i)+\Delta t[h_t(i+1)+h_t(i-1)-2h_t(i)] + \frac{\lambda\Delta t}{2}\left[\frac{h_t(i+1)-h_t(i-1)}{2}\right]^2+\sqrt{\Delta t}\,\eta_t(i)\ . \tag{6}$$

Small Δt is needed to obtain good convergence, and we choose the appropriate time step by verifying that smaller times steps do not change our results. We obtain the exponent β from $w(L,t)$ defined in Eq. (2b), since $w\sim t^{\beta}$ for $\Delta t\ll t\ll t_x$ [15].

We start with the case $\lambda=0$ (no nonlinearity) for which z and β can be found exactly from dimensional analysis [8,16]: a change of scale $x\to bx$ and $t\to b^z t$ implies $h\to b^{\alpha}h$ and

$$\eta(x,t)\to b^{\rho-1/2-z/2}\eta(x,t)$$

[from Eq. (3b)]. Equation (1) is scale invariant for the choice

$$z_0=2,\ \beta_0=\frac{1}{4}+\frac{\rho}{2}\ . \tag{7}$$

Our numerical simulation for $\lambda=0$ confirms (7).

When $\lambda\neq0$ the exponents change. Figure 1 is a log-log plot of $w(t)$ vs t for various values of λ with $\rho=\frac{1}{4}$. The inset shows the successive slopes (successive approximations to β). The exponent β approaches the same value for nonzero λ. Since changing λ should not change the universality class, we carry out our simulation for that value of λ which gives the fastest convergence to the correct value of β; then we vary the parameter ρ. The results are shown in Fig. 2. The solid, dashed, and dotted lines are the predictions from three theories [Eqs. (4a),

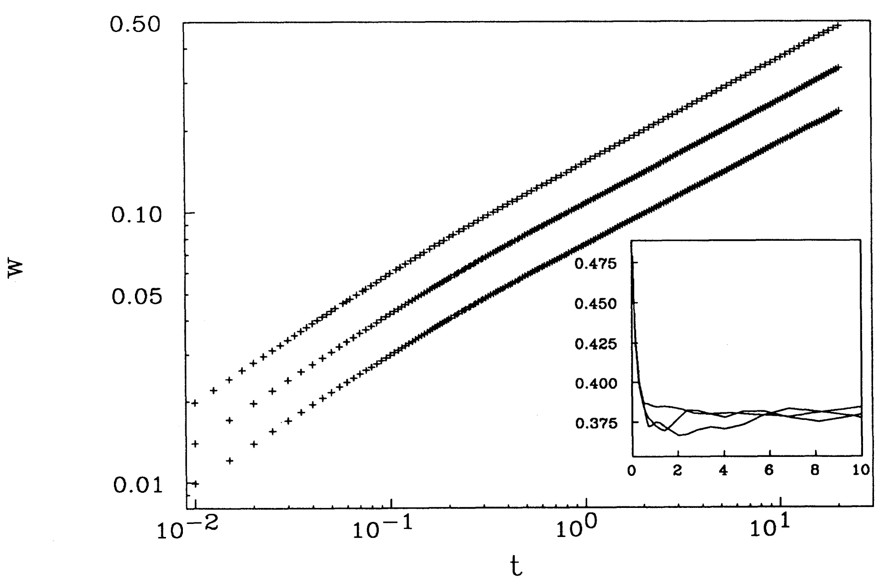

FIG. 1. Log-log plot of $w(L,t)$ vs t for $\lambda=10$, 20, 40 (bottom to top) with $\rho=\frac{1}{4}$ and time step $\Delta t=0.01$, 0.005, 0.0025, respectively. Here, $L=8192$. For clarity, each curve is shifted by $\log_{10}\sqrt{2}$. The inset shows a linear plot of successive slopes (successive approximations to β) vs time.

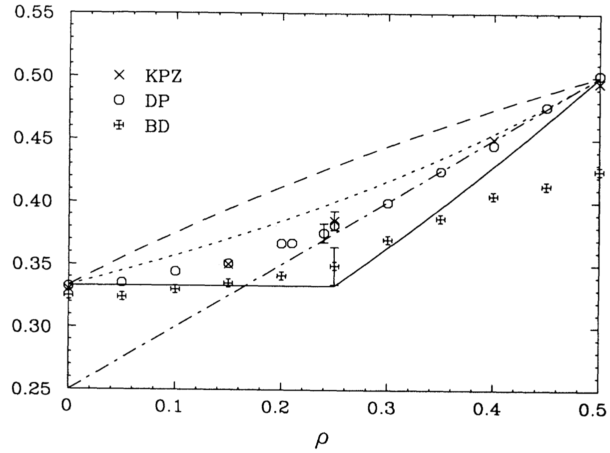

FIG. 2. Comparison of our numerical results and theoretical predictions of (4a), (4b), and (4c) (solid, dashed, and dotted lines, respectively). Typical error bars are shown for each of the three models treated. The dot-dashed line, Eq. (7), is obtained by neglecting the nonlinear term in Eq. (1).

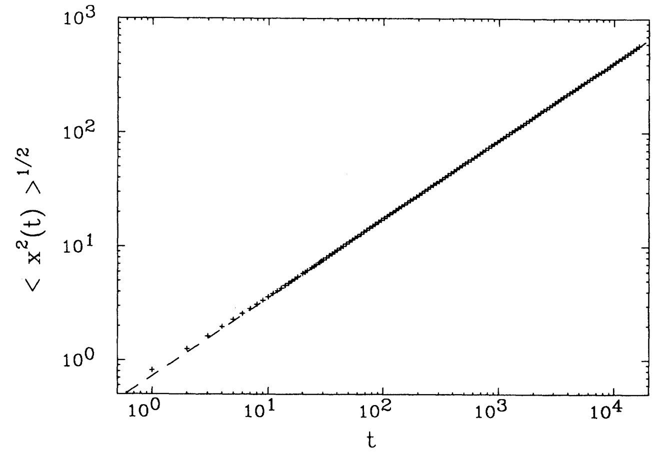

FIG. 3. Log-log plot of $\langle x^2(t)\rangle^{1/2}$ vs t for the DP problem with $\rho=0.24$. The straight line, plotted for visual guidance, has slope 0.688.

(4b), and (4c)], respectively. The numerical results agree better with (4c) than with (4a) and (4b).

To check our results, we also study the DP growth. By a simple transformation $W(\mathbf{x},t)\equiv\exp[(\lambda/2)h(\mathbf{x},t)]$, we obtain from (1)

$$\frac{\partial W}{\partial t}=\nabla^2 W+\frac{\lambda}{2}\eta(\mathbf{x},t)W\,. \tag{8}$$

Here W is the sum of Boltzmann weights for all configurations of a DP connecting (0,0) and $(\mathbf{x},t)$, and $\eta(\mathbf{x},t)$ is the potential field.

The discretized model in 1+1 dimension can be constructed on a square lattice, with the transverse direction labeled x and the longitudinal direction labeled t. Equation (3b) implies that there is no correlation along the t direction, while along the x direction the potential field is algebraically correlated. The Boltzmann weight for all paths joining the points (0,0) and (x,t) is

$$W(x,t)\equiv\sum_c \exp(-E_c/kT)\,. \tag{9}$$

Here E_c is the sum of the potential field η on configuration c, and the sum is over all configurations joining the two end points (0,0) and (x,t).

The typical transverse fluctuation scales with the length of the polymer t as $\langle x^2(t)\rangle^{1/2}\sim t^{\nu}$. At zero temperature, only the optimal path (configuration with minimum energy) makes a contribution. Since the optimal path still dominates at finite but low temperature, we choose $T=0$ to simplify our numerical task. We generate a representation of $\eta(x,t)$ [obeying Eq. (3b)], and record the end point of the optimal path $x(t)$. We average over many realizations (typically 10^5) of $\eta(x,t)$. A log-log plot of $\langle x^2(t)\rangle^{1/2}$ vs t for $\rho=0.24$ shows an excellent straight line for $t>20$ (Fig. 3). The exponent ν is related to the dynamic exponent $z=\alpha/\beta=2-\alpha$ of KPZ equation via $\nu=1/z$. Hence, to compare with the KPZ results, we define $\beta_{DP}\equiv 2\nu-1$ and show the results in Fig. 2. The agreement with our numerical results for the KPZ equation provides an excellent consistency check on our numerical methods.

(ii) Next we study the BD model with algebraically spatial correlated noise. For *uncorrelated* BD [1–4], particles rain down vertically onto the substrate until they reach one of the growth sites. A growth site is defined as the highest site on each column that belongs to the nearest neighbors of the deposition surface. Once the particles reach the growth site they stop and become a part of the deposit. Note that the deposition rule defined above allows lateral growth, while it is believed to be described by the nonlinear term $(\nabla h)^2$ in Eq. (1).

We used two methods of introducing correlated noise. *Method 1*: First we generate a representation of spatially correlated noise, which gives the set of growth probabilities $\{\eta_i\}$ for the L growth sites (there is only one growth site per column). One growth site i is randomly chosen to be a possible site for deposition. A particle is either deposited with growth probability η_i at that site or the growth

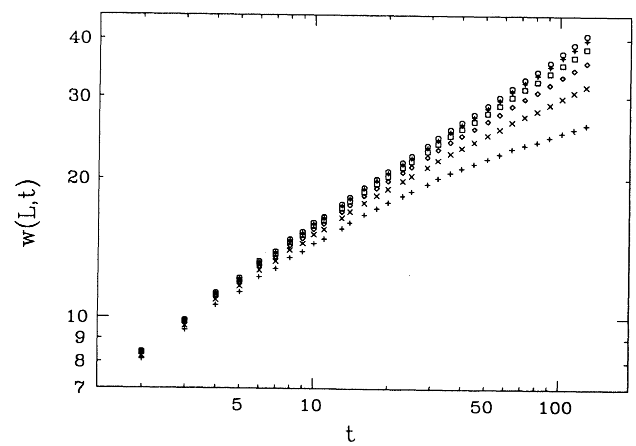

FIG. 4. Log-log plot for $w(L,t)$ vs time for $L=256$, 512, . . . ,8192 (bottom to top) for the BD model with spatially correlated noise ($\rho=\frac{1}{4}$).

site i is rejected. We used "noise reduction"—only after s particles per column are deposited, we generate a new set of growth probabilities $\{\eta_i\}$ (and increase the time by one unit) [17]. *Method 2*: At site i, we deposit a rod of length l_i proportional to the noise η_i at that site; therefore, the l_i are spatially correlated [18]. Results from both methods agree with each other.

Figure 4 shows a log-log plot of $w(t)$ vs t for $\rho=\frac{1}{4}$, and the straight line gives the exponent β for a finite system of size L. We calculate the width for systems of varying length L (from 2^8 to 2^{13}); the actual value of β is obtained by the extrapolation $L\rightarrow\infty$. This analysis was succeeded in obtaining a reliable value of $\beta(\approx\frac{1}{3})$ for $\rho=0$ (the uncorrelated noise). We then applied the same analysis with correlated noise for many values of ρ and the results are shown in Fig. 2. We find significant differences between exponents obtained from the BD model and the DP growth (or the KPZ model).

To summarize: (a) Our results for the KPZ and DP models are consistent with each other and agree better with some theoretical prediction [Eq. (4c)] than with others [Eqs. (4a) and (4b)]. For large values of the correlation parameter ρ, the exponents are compatible with those obtained from the KPZ equation in the absence of a nonlinear term [Eq. (7)], and there is effectively no contribution from the nonlinear term in (1) to the scaling exponent β. (b) Contrary to the general belief that BD is described by the KPZ equation, we find that BD with correlated noise belongs to a *different universality class* than the KPZ equation.

Note added. After this work was completed, F. Family kindly sent us a copy of Ref. [19] prior to publication. This article considers one of the models we treated (BD) and finds agreement with (4a) over the range $0\leq\rho\leq0.43$.

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[14] We find that we must transform 2^{20} numbers in order to assure the power-law correlation of (3b) is maintained within the size of the system studied (of the order 10^3).

[15] We also calculate $G(t)\equiv\langle|h(r,t)-h(0,t)|\rangle\sim t^{\beta}(\Delta t\ll t\ll t_x)$, where the average is over $L/2<r<L$. The results are the same for both methods.

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