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# Strong localization in the Anderson model with self-affine potential landscapes

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**Abstract.** We investigate the localization behavior of electronic wave functions in a one-dimensional Anderson model with self-affine potentials $\epsilon_n$. Using the transfer-matrix method, we find very large fluctuations in the localization lengths. The eigenstates are strongly localized in a way different from the states in the usual Anderson model with uncorrelated potentials, showing a patchy structure and a non-exponential decay. When the self-affine potentials are rescaled with system size to keep their overall variance fixed, an apparent transition to delocalized states occurs. The form of the corresponding localization phase diagram can be derived analytically.

**Keywords:** Anderson model, self-affine potential, strong localization
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In the past decades, the problem of localization in disordered systems has attracted much attention, for reviews see, e.g. [1, 2]. Here we investigate, how localization in one-dimensional systems is changed, when the disorder is strongly spatially correlated [3, 4, 5, 6]. We consider single-particle electronic wave functions in the tight-binding approximation, where the Schrödinger equation becomes

$$E\psi_n = \epsilon_n\psi_n - V_{n,n-1}\psi_{n-1} - V_{n,n+1}\psi_{n+1}. \tag{1}$$

Here, $E$ is the energy eigenvalue, $|\psi_n|^2$ is the probability to find an electron at site $n$, $\epsilon_n$ are the site potentials with $\langle\epsilon_n\rangle = 0$ and $V_{n,n-1}$, $V_{n,n+1}$ the hopping terms. In the following, we concentrate on the Anderson model with diagonal disorder ($V_{n,n-1} = V_{n,n+1} = V$), where we consider quantum particles in a self-affine potential landscape.

In a self-affine potential landscape, the potential at site $n+1$ depends on the potential at site $n$ by $\epsilon_{n+1} = \epsilon_n + \delta_n$, where $\delta_n$ is a random number in an interval of width $\Delta$: $-\frac{\Delta}{2} \le \delta_n \le \frac{\Delta}{2}$. If the $\delta_n$ are uncorrelated, the $\epsilon_n$ are essentially constructed by the trace of a random walk ($\epsilon_n$ corresponds to the displacement of a random walker after $n$ steps). Since the mean-square displacement $\langle r^2(t)\rangle$ at time $t$ obeys Ficks law, $\langle r^2(t)\rangle \sim t$, we have $\langle(\epsilon_{n+\ell} - \epsilon_n)^2\rangle \sim \ell$. If the $\delta_n$ are long-range correlated with a correlation function $\langle\delta_{n+\ell}\delta_n\rangle$ that decays by a power-law, $\langle\delta_{n+\ell}\delta_n\rangle \sim \ell^{-\gamma}$, $0 < \gamma < 1$, the $\epsilon_n$ correspond to the trace of a fractional random walk (see e.g. [7]).

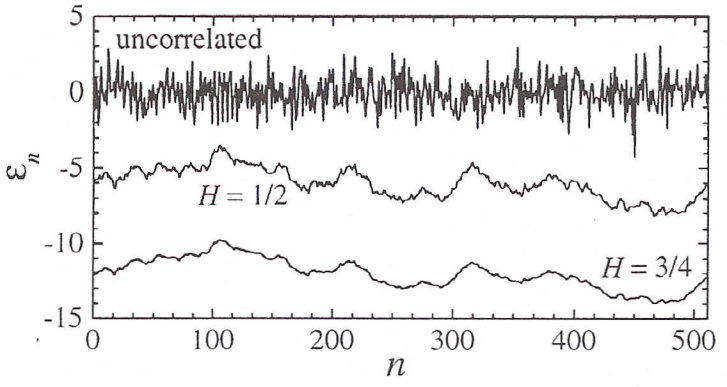


**Fig. 1** Illustration of several normalized potential landscapes. Three types of local potentials $\epsilon_n$ are shown: Uncorrelated random potential (top line) and self-affine potential landscapes with $H = 1/2$ and $3/4$. The correlated potential landscapes are shifted by multiples of 6.

where $\langle r^2(t)\rangle \sim t^{2H}$ with the Hurst exponent $H = 1 - \gamma/2$. In this case, we have

$$\langle(\epsilon_{n+\ell} - \epsilon_n)^2\rangle \sim \ell^{2H}. \tag{2}$$

Figure 1 shows, for illustration, potential landscapes for uncorrelated systems (Anderson model) and self-affine systems with $H = \frac{1}{2}$ and $H = \frac{3}{4}$. The self-affine potentials can be generated by Fourier transform, see e.g. [7].

We have used the transfer-matrix method [1] to examine the localization behavior of the electrons in the self-affine potentials. Figure 2(a) shows, for $H = \frac{3}{4}$, the localization lengths $\lambda(E)$ for several system sizes $L$. For small $L$ and $E$ near the band center, $\lambda(E)$ increases with $L$, but slower than linearly in $L$. This indicates weakly localized states. For large $L$ and at the band edges, however, this behavior is reversed: $\lambda(E)$ decreases drastically with increasing $L$, indicating *strongly* localized states.

The crossover from weakly to strongly localized states is accompanied by large fluctuations of the localization length. For obtaining the typical value $\lambda_{\rm typ}$, we averaged the localization lengths $\lambda^{(\nu)}$ logarithmically over $N = 1000$ configurations $\nu$,

$$\lambda_{\rm typ} \equiv \exp\left[\frac{1}{N}\sum_{\nu=1}^{N} \ln \lambda^{(\nu)}\right]. \tag{3}$$

The fluctuations of the localization lengths $\lambda^{(\nu)}$ can be quantified by calculating $\sigma_\lambda \equiv \exp\{[\frac{1}{N}\sum_{\nu=1}^{N} \ln^2 \lambda^{(\nu)} - (\frac{1}{N}\sum_{\nu=1}^{N} \ln \lambda^{(\nu)})^2]^{1/2}\}$. Figure 2(b) shows, for the same configurations as in (a), the fluctuations $\sigma_\lambda$ of the localization lengths $\lambda^{(\nu)}$. It can be clearly seen, that the fluctuations become extremely large at the crossover, which begins at the band edges for small system sizes $L$ and moves to the band center for larger $L$. For very large system sizes, all states are strongly localized and the fluctuations of the $\lambda^{(\nu)}$ drop to very low values again.

In order to determine the origin of the strongly localized states, we studied the shape of the eigenfunctions for $E$ at the band center [8]. We find that their amplitude

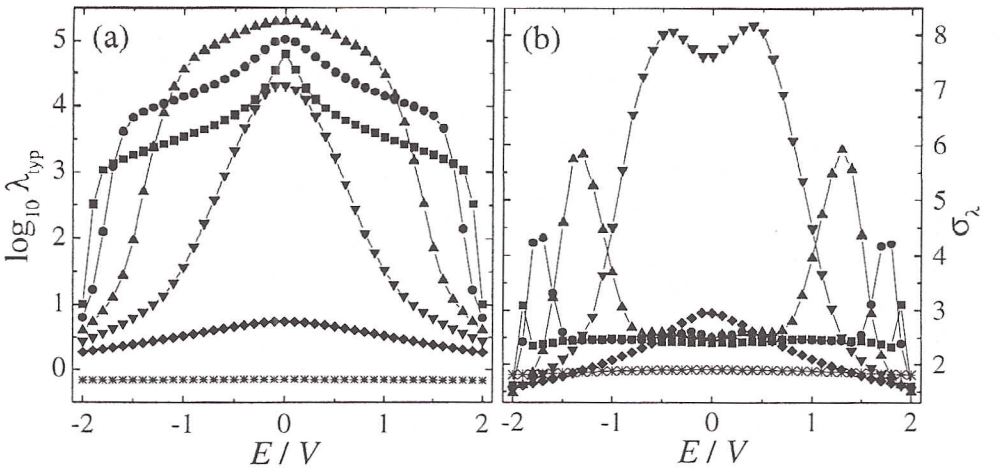


**Fig. 2** Plot of (a) the typical localization lengths $\lambda_{typ}$ and (b) the fluctuations $\sigma_\lambda$ of $\lambda$ versus energy $E$ for self-affine potential landscapes with $H = 3/4$ and $\Delta = 0.007$. The symbols indicate the system sizes $L = 2^{10}$ (boxes), $2^{12}$ (discs), $2^{14}$ (triangles up), $2^{16}$ (triangles down), $2^{18}$ (diamonds), and $2^{20}$ (stars). The averages were taken over 1000 systems.

drops sharply at those sites where $|\epsilon_n|$ exceeds $2V$. If the size of the system increases, the fraction of sites with potentials exceeding the bound $|\epsilon_n| = 2V$ increases and the wave functions become more strongly localized. Hence, in an infinite system only strongly localized states can occur. For wave functions with eigenenergy $E \neq 0$ the bounds for strong localization become $|\epsilon_n - E| = 2V$, and the crossover is reached already for smaller system sizes. These considerations are valid for all self-affine potential landscapes with $H > 0$. But the "critical" system size, where for the first time the potential of one site exceeds the bound, depends on $H$.

The strong localization behavior for self-affine potentials that we observe here has to be distinguished from the usual Anderson localization for uncorrelated potentials. In the case of Anderson localization, the wave functions have an irregular structure and their amplitudes decay roughly exponentially. In contrast, in the case of strong localization discussed here, the wave functions have a patchy structure, regions of strong and weak localization alternate, and the wave functions decay in a non-exponential manner. The non-exponential decay is the origin of the large fluctuations observed in the localization lengths calculated with the transfer-matrix method.

Finally, let us consider rescaled potential landscapes where the variance $v \equiv \langle \epsilon_n^2 \rangle - \langle \epsilon_n \rangle^2 \equiv \frac{1}{L} \sum_{n=1}^{L} \epsilon_n^2 - (\frac{1}{L} \sum_{n=1}^{L} \epsilon_n)^2$ is kept constant, independent of the system size. It is obvious from the discussions above, that for sufficiently large values of $v$ all states are strongly localized. For sufficiently small values of $v$ (and sufficiently large values of $H$), on the other hand, the fluctuations on small length scales of the rescaled potentials decrease drastically with increasing system size, and we can expect "apparent extended" states, where, as a consequence of the rescaling, $\lambda$ increases linearly with $L$ [3, 8]. Figure 3 shows the resulting phase diagram in the $E$-$H$ plane for several

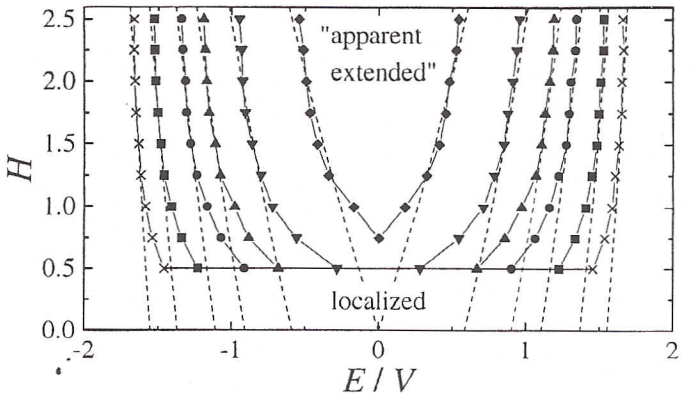


**Fig. 3** Phase diagram for the Anderson model with rescaled self-affine potentials. The transition from "apparent extended" states to localized states is shown in the $H$-$E$ plane for six disorder strengths, $v = 0.05$ (crosses), 0.1 (boxes), 0.2 (discs), 0.3 (triangles up) 0.5 (triangles down), 1.0 (diamonds). The theoretical curves, Eq. (4) with $A = 1.15$ and $B = 2$, are included in the figure for each $v$ (dashed lines). For $H < 1/2$, "apparent extended" states do not appear.

values of $v$. The critical lines have been obtained by transfer-matrix calculations, by investigating the fluctuations of the localization length that show a maximum at the transition [9]. While all states are localized for $H < 1/2$, they become apparently extended for $H > 1/2$ near the band center. The critical lines are axial symmetric to $E = 0$ and the width of the regime of "apparent extended" states decreases with increasing variance $v$. The critical lines can be estimated analytically using random walk theory [8],

$$E_c = \pm(2 - \sqrt{v}B/A^H). \tag{4}$$

Figure 3 shows, that this simple relationship describes surprisingly well the dependence of $E_c$ both, on the variance $v$ and the Hurst exponent $H$.

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