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COMMENT

Percolation thresholds on finitely ramified fractals

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Abstract. Exact renormalisation group recursion relations are used to estimate the effective percolation thresholds for site and bond percolation on finite-generation Sierpinski gaskets, and for bond percolation on branching Koch curves.

The Sierpinski gasket (SG) is a prototype of a finitely ramified fractal [1] which often served as a theoretical 'laboratory' for concepts related to fractals. In particular, Gefen *et al* [1] were the first to treat percolation on a SG, using an approximate renormalisation group (RG) recursion relation. They found that $p_c = 1$, a result which is intuitively plausible given the low connectedness of the SG. More precisely, let us look at finite-generation approximations of a SG, and let us call R_n the probability that on an nth generation SG all corners are connected. We then define an effective threshold $p_c^{(n)}$ by requiring $R_n(p = p_c^{(n)}) = c$, with $0 < c < 1$. In [1] it was found for bond percolation that

$$p_c^{(n)} \approx 1 - 1/2\sqrt{n} \qquad \text{for } n \to \infty. \tag{1}$$

The site percolation problem has been studied more recently by Yu and Yao [2], who found $p_c^{(n)} \approx 1 - 1/n$ by means of heuristic arguments and numerical simulations. Related to these problems are other transport problems on the SG, treated in [3-6].

It is the purpose of this comment to point out that for percolation on a SG one can give the *exact* RG recursion relations, similar to those given in [3] for the problem of Joule heat distribution on a SG, and in [5, 6] for self-avoiding walks and trails.

In addition to the probability R_n for percolation from any corner to both others, we need the probability for percolation between two corners, but not between them and the third. We call this S_n. Obviously, $1 - R_n - 3S_n$ is the probability that there is no percolation between any pair of corners. Graphically, we represent R_n and S_n as shown in figure 1.

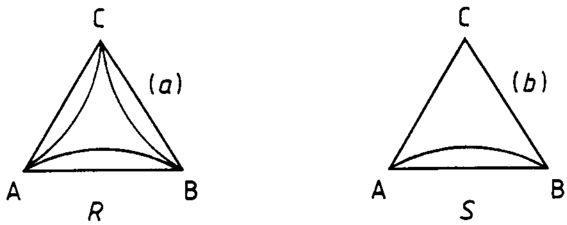

Figure 1. Probabilities for a finite-generation Sierpinski gasket to percolate: (*a*) from any corner to any other corners; (*b*) from corner A to corner B, but not to corner C.

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For bond percolation, the RG recursion for R_n is shown graphically in figure 2. Together with the somewhat more complicated recursion for S_n, we then obtain the exact relations

$$\begin{aligned} R_{n+1} &= R_n^3 + 6R_n^2 S_n + 3R_n S_n^2 \\ S_{n+1} &= (R_n + S_n)^2 - 4R_n^2 S_n + S_n^3 - R_n^3. \end{aligned} \tag{2}$$

We make now an ansatz

$$\begin{aligned} R_n &= 1 + \alpha/n + \mathrm{O}(n^{-3}) \\ S_n &= \beta/n + \gamma/n^2 + \mathrm{O}(n^{-3}) \end{aligned} \tag{3}$$

with open parameters α, β and γ. Notice that no term $\sim 1/n^2$ appears in the ansatz for R_n, as such a term can always be absorbed in the term $\sim 1/n$ by a translation $n \to n + \text{constant}$. The recursion relations give the unique solution

$$\alpha = \tfrac{3}{4} \qquad \beta = -\tfrac{1}{4} \qquad \gamma = -\tfrac{1}{16}. \tag{4}$$

In order to have non-negative probabilities, we can use this solution only for $n < 0$. Level $n = 0$ corresponds to the outer length scale. Assume now that the recursions (2) hold only for $n > -N$, i.e. level $n = -N$ corresponds to the inner length scale. At this scale, we have a simple triangle with bond probability p, i.e.

$$\begin{aligned} R_{-N} &= p^3 + 3p^2(1-p) \\ S_{-N} &= p(1-p)^2. \end{aligned} \tag{5}$$

Comparing (3) and (5) gives then in agreement with [1]

$$p_c^{(N)} = 1 - 1/2\sqrt{N} + O(N^{-1}) \qquad \text{(bond percolation).} \tag{6}$$

This result is supported by numerical simulations which were performed using a technique described in detail in [4]. The effective percolation threshold was determined according to the condition $R_n(p = p_c^{(n)}) = 0.95$, where the constant $c = 0.95$ was chosen arbitrarily.

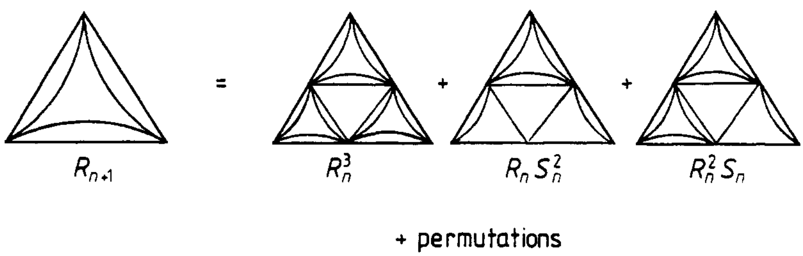

Figure 2. Recursion relation for R_n, the probability to percolate from any corner to any other.

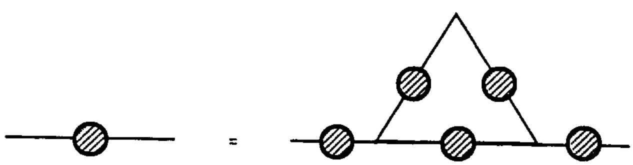

Figure 3. Recursion relation defining a branching Koch curve.

For site percolation, the recursion relations are somewhat more complicated. A straightforward analysis gives

$$\begin{aligned} R_{n+1} &= R_n^3 p^3 + 3R_n p^2((1-p)R_n^2 + 2R_nS_n + S_n^2) \\ S_{n+1} &= p[(S_n + R_n)^2 + pS_n^3 - p(3+p)R_n^2S_n - p(2-p)R_n^3]. \end{aligned} \tag{7}$$

We were not able to solve this analytically as in the bond percolation case. It is however trivial to iterate (7) numerically, with the initial values for R and S given by (5). From such iterations, we found

$$p_c^{(N)} \approx 1 - 0.5/N \qquad \text{(site percolation)} \tag{8}$$

which agrees qualitatively but not quantitatively with the result of [2]. We might add that we also performed numerical iterations on (2), thereby verifying (3)-(6).

Finally, we should mention that similar (and indeed simpler) exact recursion relations can be given for many other fractals, including in particular branching Koch curves [7]. In the latter case, one finds in general an exponential convergence of p_c towards 1. For instance, for bond percolation on the branching Koch curve shown in figure 3 we get a RG relation for the probability R_n of percolation

$$R_{n+1} = R_n^3(1 - R_n) \tag{9}$$

from which we obtain $p_c^{(N)} \approx 1 - \text{constant}/2^N$. Again, this result is found to be in perfect agreement with numerical simulations.

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