Conductivity in hierarchical networks with a broad distribution of resistors

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We study transport in hierarchical structures in which the probability density function (PDF) of bond conductances is a stable law. We demonstrate that the singular part of the PDF remains invariant in each iteration step. Thus the leading contribution to the conductivity exponent comes from singly connected bonds.

In a recent paper, Machta, Guyer, and Moore[1] studied a hierarchical resistor network model, where the conductances σ were chosen from a power-law distribution which has the property

$$p(\sigma)\sim\sigma^{-\alpha},\ 0<\sigma<1\ , \tag{1}$$

as $\sigma\rightarrow 0$. Their analysis was done both by theoretical and numerical means. In their theory they applied renormalization-group theory using an ansatz about the form of the resistance of a blob of order n (see Fig. 1). Their approach leads to the following results for the conductivity exponents:

$$\begin{aligned} t\sim t_0,\ & t_0>(d-2)+1/(1-\alpha) \\ \sim(d-2)+1/(1-\alpha),\ & t_0<(d-2)+1/(1-\alpha)\ , \end{aligned} \tag{2}$$

where t_0 is the resistivity exponent of a network composed of equal resistors, and t_0 can be expressed in terms of detailed parameters of the model.[1] These results were supported by simulated data.

In this Comment we provide a rigorous proof that the singular part of the probability distribution function (PDF) remains invariant under the renormalization transformation, an assumption needed to derive Eq. (2). This is done by solving for the singular parts of the PDF under the nonlinear transformation of the total resistivity going from the nth to the $(n+1)$st generation [cf. Eqs. (2.8) and (2.9) in Ref. 1].

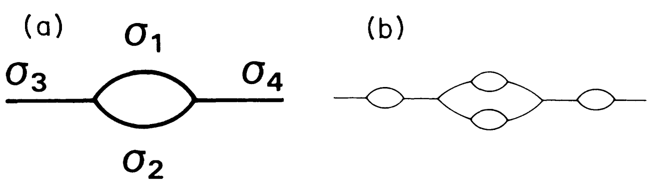

FIG. 1. The hierarchical structure studied in the paper is illustrated. (a) shows the first generation of the fractal and (b) the second generation. An $(n+1)$-generation blob is formed from two parallel n-generation chains. An $(n+1)$ chain is composed of an $(n+1)$ blob in series with two n chains.

For simplicity we consider the case of the simple hierarchical structure shown in Fig. 1 but the proof is easily generalized. We start with the first generation and we denote the four conductances as σ_1, σ_2, σ_3, and σ_4 (see Fig. 1). The distribution of these conductances is chosen from a distribution with the property given in Eq. (1). The total conductivity for a specific configuration in this iteration is given by

$$\sigma=[1/(\sigma_1+\sigma_2)+1/\sigma_3+1/\sigma_4]^{-1}\ . \tag{3}$$

The behavior in this model that dominates the conductivity results from the analytical behavior of $P(\sigma)$ for $\sigma\sim 0$. Equation (1) gives that behavior for a single resistor. Let us drive the analogous small-σ behavior for the density of σ given in Eq. (3). The derivation requires that we calculate the separate probability densities of $1/\sigma_3+1/\sigma_4$ and $1/(\sigma_1+\sigma_2)$. We consider the former first, by letting $v=1/\sigma$. The probability density of v has the property

$$q(v)=p(\sigma)d\sigma/dv=Av^{\alpha-2},\ 1<v<\infty \tag{4}$$

for $v\gg 1$. Thus, $q(v)$ is in the domain of attraction of a stable law of order $2-\alpha$ so that the sum of two such variables is also asymptotically stable and of the same order.[2]

We next turn our attention to the sum of random variables, $\sigma'=\sigma_1+\sigma_2$. The PDF of this sum is $p_2(\sigma')$, where

$$\begin{aligned} p_2(\sigma')&\sim\int_0^{\sigma'}\sigma^{-\alpha}(\sigma'-\sigma)^{-\alpha}d\sigma \\ &=(\sigma')^{2\alpha-1}\int_0^1\Gamma^{-\alpha}(1-\Gamma)^{-\alpha}d\Gamma\ . \end{aligned} \tag{5}$$

We will need the PDF of $v'=1/\sigma'$ which by an argument similar to that given earlier yields

$$q_2(v)\sim 1/v^{3-2\alpha} \tag{6}$$

for sufficiently large v. The next step is to find the singular part of the PDF of $v+v'$, where Eqs. (4) and (6) give the singular parts of the component random variables. In order to calculate the singular part of the sum we note that the small-s behavior of the characteristic function of a non-negative random variable, whose asymptotic form is $v^{-\beta}$ for $v \gg 1$, is[2]

$$C(s)=\int_0^\infty q(v)e^{-sv}dv \sim \exp(-s^{\beta-1}) \tag{7}$$

as $s \rightarrow 0$, where we have omitted a constant in the exponent. Thus, the characteristic function for the sum of two variables with exponents β_1 and β_2 is

$$C(s)=C_1(s)C_2(s) \sim \exp(-s^{\beta_1-1}-s^{\beta_2-1}) \sim \exp(-s^{\min(\beta_1,\beta_2)-1}) \ . \tag{8}$$

In our case $\min(2-\alpha, 3-2\alpha)=2-\alpha$ so that the PDF of the sum is asymptotically $v^{\alpha-2}$. But by our earlier reasoning this result implies that the singular part of the PDF of $1/(v+v')$ has the form $p(\sigma) \sim \sigma^{-\alpha}$ as $\sigma \rightarrow 0$. Thus, the exponent appearing in the PDF of the conductivity, Eq. (1), remains invariant under the transformation (3), as was found numerically in Ref. 1.

[1] J. Machta, R. A. Guyer, and S. M. Moore, Phys. Rev. B **33**, 4818 (1986).

[2] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1971), Vol. 2.